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12. A Review on Disjunctive ideals of Almost Distributive Lattices

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In 1981, U.M. Swamy and G.C. Rao proposed the concept of an Almost Distributive Lattice (ADL), offering a unified approach to understanding various generalizations of Boolean algebras and distributive lattices. Their introduction of ideals in ADLs mirrored the corresponding concept in distributive lattices. They also proved that the collection of principal ideals in an ADL, represented by PI(L), constitutes a distributive lattice. This foundational work laid the groundwork for extending key ideas in lattice theory to ADLs.

Subsequently, in 2009, G.C. Rao and S. Ravi Kumar made significant contributions to the theory of ADLs by focusing on minimal prime ideals. They developed characterizations of normal ADLs in terms of their prime ideals, minimal prime ideals, and annihilator ideals, thereby enriching the structural understanding of these lattices.

In 2015, M. Sambasiva Rao introduced innovative concepts such as disjunctive ideals, strongly disjunctive ideals, and normal prime ideals within the context of distributive lattices. He established criteria for an ideal in a lattice to qualify as a disjunctive ideal.

This work extended to ADLs, where disjunctive ideals were similarly defined. It was noted that the collection of all disjunctive ideals in an ADL forms a complete lattice. Additionally, it was shown that for any ideal I in an ADL, its extension I^e is a disjunctive ideal containing I.

Rao further explored the properties of normal prime ideals in ADLs, providing a detailed study of their behavior. He also identified conditions under which a minimal prime ideal could be classified as a normal prime ideal. These studies culminated in the establishment of various theorems, supported by definitions and lemmas as necessary to formalize and validate the results. In this paper, L refers to the ADL that contains maximal elements, unless otherwise stated.

Definition: For each non-empty set P of L, consider $P^0 = \{s \in L / (p)^* \vee (s)^* = L$, for every $p \in P\}$

Lemma: If P, Q are any non-void subsets of L then

1. P^0 is an ideal of L.
2. $P \cap P^0 \subseteq \{0\}$
3. If $P \subseteq Q$, then $Q^0 \subseteq P^0$
4. $P \subseteq P^{00}$
5. $P^{000} = P^0$
6. $P^0 = L$ iff $P = \{0\}$.

Theorem: For any ideals P and Q of L, the following are true:

1. $(P \vee Q)^0 = P^0 \cap Q^0$
2. $(P \cap Q)^{00} \subseteq P^{00} \cap Q^{00}$
3. $P^{00} \cap Q^{00} \subseteq (P \vee Q)^{00}$
4. $P \subseteq Q^0 \Rightarrow P \cap Q = \{0\}$

Corollary: Let $\{P_i / i \in \Delta\}$ is a collection of ideals of L. Then $(\bigcap_{i\in\Delta} P_i)^{00} = \bigcap_{i\in\Delta} (P_i)^{00}$

Theorem: For any non-avoid subset P of L, we have $P^0 = \bigcap_{p\in P} (p)^0$

Lemma: For any ideal P of L, $P^0 \subseteq P^*$.

Theorem: T following conditions are equivalent for any ADL L:

1. L is normal.
2. For every ideals P, Q of L, $P \cap Q = \{0\}$ iff $P \subseteq Q^0$.
3. For any ideal P of L, $P^0 = P^*$.
4. For any $p \in L$, $(p)^0 = (p)^*$.

Definition: An ideal P in L is called disjunctive if for every p, q ϵ L, $(p)^0 = (q)^0$ and p ϵ P implies q ϵ P.

Lemma: In an ADL L, we have:

1. $(p)^0$ is disjunctive in L, for each p ϵ L.
2. If P is an ideal in L such that p ϵ L, p ϵ P implies $(p)^{00} \subseteq$ P, then P is a disjunctive.

Theorem: Let L be an ADL and S closed under meet in L. Then $P = \{p \in L \mid (p)^* \vee (s)^* = L$ for some $s \in S\}$ is disjunctive.

Lemma: If union of disjunctive ideals of L is an ideal, then it is also disjunctive.

Theorem: For any ideal P of L, consider $P^e = \{a \in L / (p)^0 \subseteq (a)^0$ for each $p \in P\}$.

Lemma: The following are equivalent in an ADL L:

1. $P \subseteq P^e$
2. $P \subseteq Q \Rightarrow P^e \subseteq Q^e$
3. $(P \cap Q)^e \subseteq P^e \cap Q^e$
4. $P^e \vee Q^e \subseteq (P \vee Q)^e$
5. $(P^0)^e = P^0$.

Theorem: For any ideal P in L, P^e is a disjunctive ideal of L containing P.

Theorem: Let P be a disjunctive ideal of an ADL L. Then $P^e = P$.

Theorem: The set $P_D(L)$ of all disjunctive ideals of L forms a complete lattice on its own.

Theorem: In an ADL L, the following conditions are equivalent:

1. Every ideal is a disjunctive ideal.
2. Every principal ideal is a disjunctive ideal.
3. Every prime ideal is a disjunctive ideal.

4. For any s, t $\in$ L, $(s)^0 = (t)^0$ implies $(s] = (t]$.

Definition: A prime ideal M of L is said to be normal if to each m $\in$ M, there is $n \notin M$ such that $(m)^0 \vee (n)^0 = L$.

Theorem: Every normal prime ideal of L is minimal.

Theorem: Every minimal prime ideal of L is normal.

Theorem: For any normal prime ideal M in L and m $\in$ L, we have $m \notin M$ iff $(m)^0 \subseteq M$.

- Dr. T. Srinivasa Rao