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12. A Review on Disjunctive ideals of Almost Distributive Lattices

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An Almost Distributive Lattice (ADL) was introduced by U.M. Swamy and G.C. Rao in 1981 as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. In 2009, G.C. Rao and S. Ravi Kumar proved some important results on minimal prime ideals of an ADL and characterized the normal ADL in terms of its prime ideals, minimal prime ideals and annihilator ideals. In 2015, M. Sambasiva Rao has introduced the concepts of disjunctive ideals, strongly disjunctive ideals and normal prime ideals in distributive Lattices. In that paper, he derived a set of equivalent conditions for every ideal of a lattice to become a disjunctive ideal.

Introduced the concept of disjunctive ideals in an ADL, analogous to that in a distributive lattice and also observed that the set of all disjunctive ideals of an ADL can be made into a complete lattice. For any ideal I of an ADL, it is noted that the extension I^e is a disjunctive ideal containing I. Introduced the concept of normal prime ideals in an ADL and studied their properties and given an equivalent condition for every minimal prime ideal to convert into a normal prime ideal.

Based on the above concepts, the following theorems are established. The definitions and lemmas are also given where ever and whenever necessary.

Definition: For any nonempty subset A of an ADL L, define $A^0 = \{x \in L / (a)^* \vee (x)^* = L$, for all $a \in A\}$

Lemma: Let L be an ADL with maximal element m. For any nonempty subset A, B of L we have:

1. A^0 is an ideal of L.
2. $A \cap A^0 \subseteq \{0\}$
3. If $A \subseteq B$, then $B^0 \subseteq A^0$
4. $A \subseteq A^{00}$
5. $A^{000} = A^0$
6. $A^0 = L$ iff $A = \{0\}$.

Theorem: Let I, J be any two ideals of ADL L. Then we have the following:

1. $(I \vee J)^0 = I^0 \cap J^0$
2. $(I \cap J)^{00} \subseteq I^{00} \cap J^{00}$
3. $I^{00}\ J^{00} \subseteq (I \vee J)^{00}$
4. $I \subseteq J^0 \Rightarrow I \cap J = \{0\}$

Corollary: Let L be an ADL with maximal elements. If $\{I_i / i \in \Delta\}$ is a family of ideals of L, then $(\bigcap_{i\in\Delta} I_i)^{00} = \bigcap_{i\in\Delta} (I_i)^{00}$.

Theorem: Let L be an ADL with maximal elements. For any nonempty subset A of L, we have $A^0 = \bigcap_{a\in A} (a)^0$.

Lemma: Let I be any ideal of an ADL L with maximal elements. Then $I^0 \subseteq I^*$.

Theorem: Let L be an ADL with maximal elements. Then the following conditions are equivalent:

1. L is a normal ADL.

2. For any ideals I, J of L, $I \cap J = \{0\}$ iff $I \subseteq J^0$.
3. For any ideal I of L, $I^0 = I^*$.
4. For any a ϵ L, $(a)^0 = (a)^*$.

Definition: An ideal I of an ADL L is said to be disjunctive if for any x, y ϵ L, $(x)^0 = (y)^0$ and x ϵ I implies y ϵ I.

Lemma: Let L be an ADL with maximal elements. Then we have the following:

1. (x)0 is a disjunctive ideal of L, for all x ϵ L.
2. If I is an ideal of L such that x ϵ L, x ϵ I implies $(x)^{00} \subseteq I$, then I is a disjunctive ideal of L.

Theorem: Let L be an ADL and S a multiplicatively closed subset of L. Then the set I = $\{x \in L \mid (x)^* \vee (a)^* = L$ for some $a \in S\}$ is a disjunctive ideal of L.

Lemma: Let L be an ADL. If the set-theoretic union of disjunctive ideals of L is an ideal, then it is also a disjunctive ideal of L.

Theorem: Let L be an ADL. For any ideal I of L, define $I^e = \{x \in L/\ (a)^0 \subseteq (x)^0$ for all $a \in I\}$.

Lemma: Let L be an ADL with maximal elements. Then we have the following:

1. $I \subseteq I^e$
2. $I \subseteq J \Rightarrow I^e \subseteq J^e$
3. $(I \cap J)^e \subseteq I^e \cap J^e$
4. $I^e \vee J^e \subseteq (I \vee J)^e$
5. $(I^0)^e = I^0$.

Theorem: Let L be and ADL with maximal elements and I an ideal of L. Then I^e is a disjunctive ideal of L containing I.

Theorem: Let I be a disjunctive ideal of an ADL L. Then $I^e = I$.

Theorem: Let L be an ADL with maximal elements. Then the set $I_D(L)$ of all disjunctive ideals of L forms a complete lattice on its own.

Theorem: Let L be an ADL with maximal elements. Then the following conditions are equivalent:

1. Every ideal is a disjunctive ideal.
2. Every principal ideal is a disjunctive ideal.
3. Every prime ideal is a disjunctive ideal.
4. For any a, b $\in$ L, $(a)^0 = (b)^0$ implies $(a] = (b]$.

Definition: Let L be an ADL with maximal elements. A prime ideal P of L is said to be normal if to each x $\in$ P, there exists $y \notin P$ such that $(x)^0 \vee (y)^0 = L$.

Theorem: Every normal prime ideal of an ADL L is a minimal prime ideal.

Theorem: Let L be an ADL with maximal elements. Then every minimal prime ideal of L is normal prime ideal.

Theorem: Let P be a normal prime ideal of an ADL L with maximal elements. Then for each x $\in$ L, we have $x \notin P$ iff $(x)^0 \subseteq P$.

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