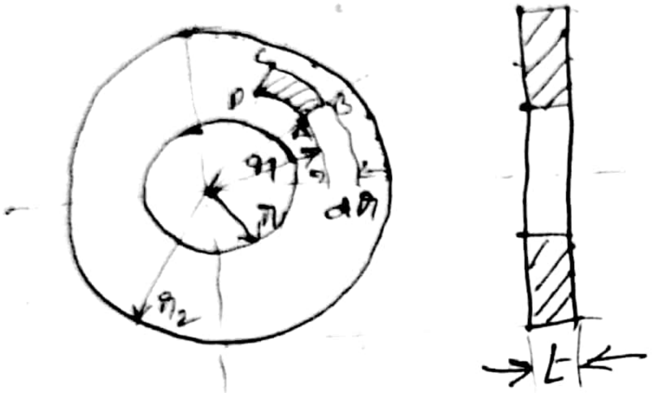
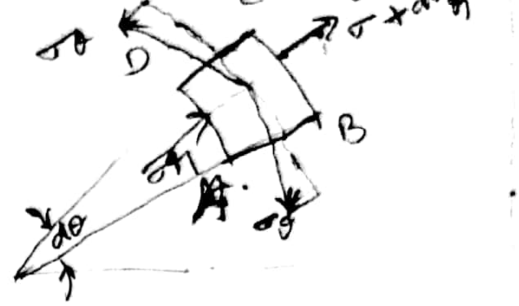
Stresses in Rotating disc :-

consider a flat disc of uniform thickness 't' rotating at speed 'ω'.

Let r_1 and r_2 be the inner and outer radius of the disc.





Rotating disc of Uniform Thickness

consider an element ABCD and stresses acting on the faces are shown in fig.

Volume of the element ABCD $= r\,d\theta \times dr \times t$

Radial force on the element due to rotation (Centrifugal force) $= \rho\,(r\,d\theta\,dr\,t)\,\omega^2 r$

$= \rho\,\omega^2 r^2 t\,d\theta\,dr$

outward force on the face BC $= (\sigma_r + d\sigma_r)(r+dr)\,d\theta \times t$

Inward force on face AD $= r\,d\theta \times t \times \sigma_r$

Inward forces on face AB & DC $= \sigma_\theta \times dr \times t \times \frac{d\theta}{2}$, $\sigma_\theta\,dr\,t\,\frac{d\theta}{2}$

and ~~$\sigma_\theta \times t\,dr$~~

Resolving the forces in the radial outward direction

$(\sigma_r + d\sigma_r)(r+dr)\,d\theta\,t - \sigma_r\,r\,d\theta\,t - \sigma_\theta \times t\,dr\,d\theta$.

Simplying and neglecting small quantities, $\Rightarrow (\sigma_r - \sigma_\theta)\,dr\,d\theta\,t + d\sigma_r\,r\,d\theta\,t$

Scanned with CamScanner

for equilibrium of the element

$$\rho(r\,d\theta\,dr\,t)\,\omega^2 r + (\sigma_r - \sigma_\theta)\,dr\,d\theta\,t + d\sigma_r\, r\,d\theta\,t = 0$$

$$\rho\,\omega^2 r^2 + (\sigma_r - \sigma_\theta) + r\frac{d\sigma_r}{dr} = 0$$

$$(\sigma_r - \sigma_\theta) = -\rho\,\omega^2 r^2 - r\frac{d\sigma_r}{dr} \quad -(1)$$

Due to rotation, let r becomes $r+u$ and

$r+dr$ becomes $r+dr+du$

Then circumferential strain ϵ_θ :

$$\epsilon_\theta = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}$$

radial strain $\epsilon_r = \dfrac{(r+dr+du) - (r+dr)}{dr} = \dfrac{du}{dr}$

Also, we know that $\epsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{u}{r}$

$$\therefore\ u = \frac{r}{E}(\sigma_\theta - \nu\sigma_r) \quad -(2)$$

and $\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{du}{dr} \quad -(3)$

Differentiating eq (2) w.r.t. r,

$$\frac{du}{dr} = \frac{r}{E}\left[\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right] + \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \quad -(4)$$

comparing Eqs (3) & (4)

$$\frac{1}{E}(\sigma_r - \nu\sigma_\theta) = \frac{r}{E}\left[\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right] + \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\therefore \frac{1}{E}\left[\sigma_r - \sigma_\theta - \nu\sigma_\theta + \nu\sigma_r\right] = \frac{r}{E}\left[\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right]$$

$$(1+\nu)(\sigma_r - \sigma_\theta) = r\left[\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right] \quad -(5)$$

substituting eq (1) in (5)

$$(1+\nu)\left[-r\frac{d\sigma_r}{dr} - \rho\omega^2 r^2\right] = r\left[\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right]$$

$$(1+\nu)\left[-r\frac{d\sigma_r}{dr}\right] - (1+\nu)\rho\omega^2 r^2 = r\frac{d\sigma_\theta}{dr} - r\nu\frac{d\sigma_r}{dr}$$

$$-r\frac{d\sigma_r}{dr} - \cancel{\nu r\frac{d\sigma_r}{dr}} - (1+\nu)\rho\omega^2 r^2 - r\frac{d\sigma_\theta}{dr} + \cancel{r\nu\frac{d\sigma_r}{dr}} = 0$$

$$-r\left[\frac{d\sigma_r}{dr} + \frac{d\sigma_\theta}{dr}\right] - (1+\nu)\rho\omega^2 r^2 = 0$$

$$\frac{d}{dr}\left[\sigma_r + \sigma_\theta\right] + (1+\nu)\rho\omega^2 r = 0$$

$$\Rightarrow \frac{d}{dr}(\sigma_r + \sigma_\theta) + (1+\nu)\rho\omega^2 r) = 0 \quad -(6)$$

Integrating, we get

$$\sigma_r + \sigma_\theta = \frac{1}{2}(1+\nu)\rho\omega^2 r^2 = C_1 \quad -(7)$$

where C_1 = constant of integration

$$\therefore \sigma_\theta = C_1 - \sigma_r - \frac{1}{2}(1+\nu)\rho\omega^2 r^2$$

Substituting in eq (a),

$$2\sigma_r + r\frac{d\sigma_r}{dr} = C_1 - \left(\frac{3+\nu}{2}\right)\rho\omega^2 r^2$$

Multiplying each side by r

$$2r\sigma_r + r^2\frac{d\sigma_r}{dr} = C_1 r - \left(\frac{3+\nu}{2}\right)\rho\omega^2 r^3$$

$$\frac{d}{dr}(r^2\sigma_r) = C_1 r - \left(\frac{3+\nu}{2}\right)\rho\omega^2 r^3$$

Integrating the above eq.

$$r^2\sigma_r = \frac{C_1 r^2}{2} - \left(\frac{3+\nu}{2}\right)\frac{\rho\omega^2 r^4}{4} + C_2$$

$$\therefore \sigma_r = \frac{C_1}{2} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r^2 + \frac{C_2}{r^2} \quad \text{--- (8)}$$

where C_2 is another constant of integration

Substituting eq (8) in (7)

$$\sigma_\theta = \frac{C_1}{2} - \frac{C_2}{r^2} - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2 \quad \text{--- (9)}$$

and $$u = \frac{r}{E}\left[(1-\nu)\frac{C_1}{2} - (1+\nu)\frac{C_2}{r^2} - \left(\frac{1-\nu^2}{8}\right)\rho\omega^2 r^2\right] \quad \text{--- (10)}$$

C_1 & C_2 can be determined from the boundary conditions.

Consider:-

$$\sigma_r = \frac{c_1}{2} + \frac{c_2}{r^2} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r^2$$

$$\sigma_\theta = \frac{c_1}{2} - \frac{c_2}{r^2} - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2$$

Case:-1 Solid disc

At $r=0$, $u=0$ and σ_r & σ_θ are infinite

$\therefore$ $c_2 = 0$

$$\therefore \sigma_r = \frac{c_1}{2} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r^2 \;\&\; \sigma_\theta = \frac{c_1}{2} - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2$$

at $r = r_2$ $\quad \sigma_r = 0$

$$\therefore 0 = \frac{c_1}{2} - \left(\frac{3+\nu}{8}\right)\rho\omega^2 r_2^2$$

$$\therefore c_1 = \left(\frac{3+\nu}{4}\right)\rho\omega^2 r_2^2$$

$$\therefore \sigma_r = \left(\frac{3+\nu}{8}\right)\rho\omega^2 (r_2^2 - r^2)$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho\omega^2 r_2^2 - \left(\frac{1+3\nu}{8}\right)\rho\omega^2 r^2$$

The values of σ_r and σ_θ are Maxim at

$r = 0$

$$(\sigma_r)_{max} = \left(\frac{3+\nu}{8}\right)\rho\omega^2 r_2^2 \quad \text{and}$$

$$(\sigma_\theta)_{max} = \left(\frac{3+\nu}{8}\right)\rho\omega^2 r_2^2$$

Radial displacement of the disc is

$$u = \frac{r}{E}(\sigma_\theta - \nu \sigma_r)$$

Case-2 Hollow Disc

$$\sigma_r = \frac{c_1}{2} + \frac{c_2}{r^2} - \left(\frac{3+\nu}{8}\right) \rho \omega^2 r^2$$

$$\sigma_\theta = \frac{c_1}{2} - \frac{c_2}{r^2} - \left(\frac{1+3\nu}{8}\right) \rho \omega^2 r^2$$

at $r = r_1$ $\sigma_r = 0$ and at $r = r_2$, $\sigma_r = 0$

$$0 = \frac{c_1}{2} + \frac{c_2}{r_1^2} - \left(\frac{3+\nu}{8}\right) \rho \omega^2 r_1^2$$

$$0 = \frac{c_1}{2} - \frac{c_2}{r_2^2} - \left(\frac{3+\nu}{8}\right) \rho \omega^2 r_2^2$$

Solving for c_1 and c_2

$$c_1 = \left(\frac{3+\nu}{4}\right) \rho \omega^2 (r_1^2 + r_2^2)$$

$$c_2 = -\left(\frac{3+\nu}{4}\right) \rho \omega^2 r_1^2 r_2^2$$

$$\therefore \sigma_r = \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[r_1^2 + r_2^2 - r^2 - \frac{r_1^2 r_2^2}{r^2}\right] \quad \text{—(a)}$$

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left[r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \left(\frac{1+3\nu}{3+\nu}\right) r^2\right] \quad \text{—(b)}$$

σ_θ is Maximum at $r = r_1$

$$(\sigma_\theta)_{Max} = \left(\frac{3+\nu}{4}\right) \rho\omega^2 \left[r_2^2 + \left(\frac{1-\nu}{3+\nu}\right) r_1^2\right] \quad \text{--- (6)}$$

for σ_r to be Maximum

$$\frac{d\sigma_r}{dr} = 0$$

from eq (a)

$$2r - \frac{2r_1^2 r_2^2}{r^3} = 0$$

$$r = \sqrt{r_1 r_2}$$

$$(\sigma_r)_{max} = \left(\frac{3+\nu}{8}\right) \rho\omega^2 (r_2 - r_1)^2$$

When $r_1 \to 0$ i.e a very small hole is drilled in the disc

then

$$(\sigma_\theta)_{max} = \left(\frac{3+\nu}{4}\right) \rho\omega^2 r_2^2$$

∴ The Max. Hoop stress in a rotating disc is twice as large when there is a small hole at its axis of rotation as that when the disc is solid.

Prob ① A circular thin disc of outter radius 20cm and inner radius 5 cm is rotating at 3000rpm. Determine the variation of radial and hoop stresses in the disc. $\rho = 7800 kg/m^3$ $\nu = 0.3$.

Sol
$$\sigma_r = \left(\frac{3+\nu}{8}\right) \rho \omega^2 \left(r_1^2 + r_2^2 - r^2 - \frac{r_1^2 r_2^2}{r^2}\right)$$

$r_1 = 5cm = 0.05m$ $r_1 = 0.2m$

$N = 3000rpm$ $\omega = \frac{2\pi \times 3000}{60}$

$\rho = 7800 kg/m^3$ $\nu = 0.3$

$$\sigma_r = \left(\frac{3+0.3}{8}\right) 7800 \times \left(\frac{2\pi \times 3000}{60}\right)^2 \times \left[0.05^2 + 0.2^2 - r^2 - \frac{0.05^2 \times 0.2^2}{r^2}\right]$$

Also σ_r is max on at $r = \sqrt{r_1 r_2} = \sqrt{5 \times 20} = 10cm$
$= 0.1m$.

$\therefore$ σ_r values at different r values is tabulated as shown as

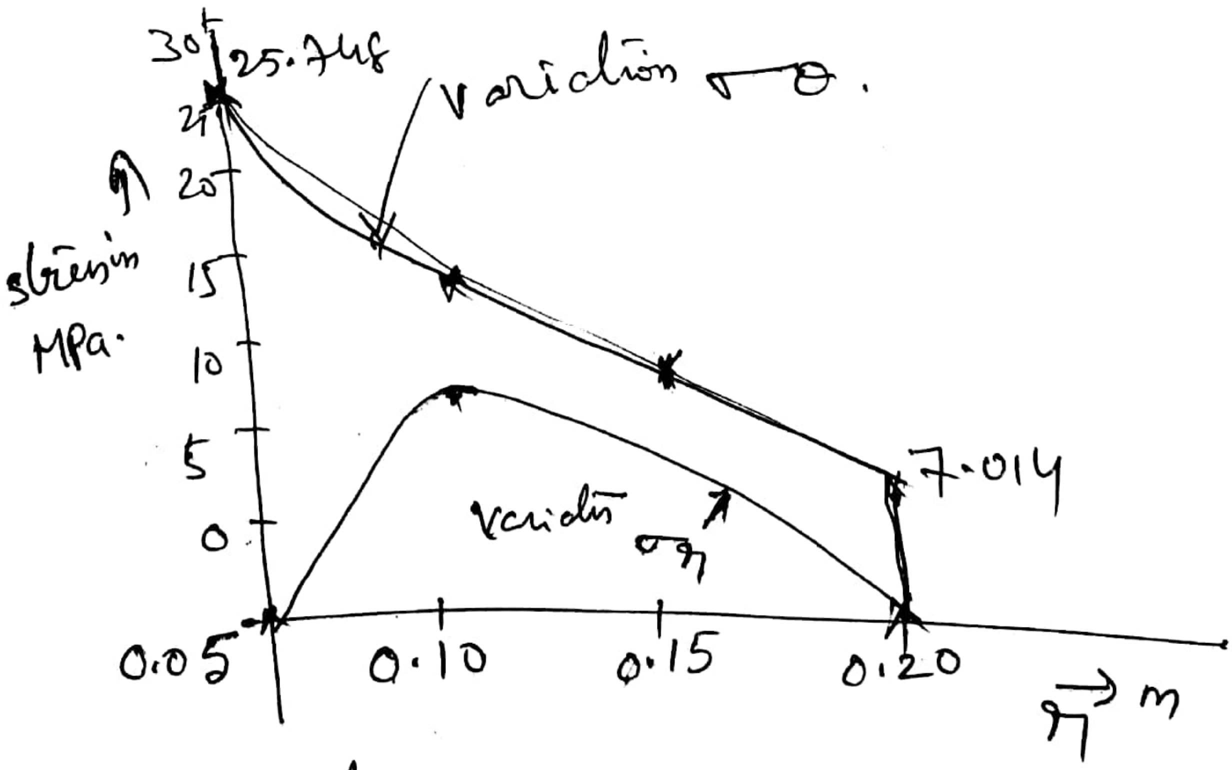
r cm	0.05	0.10	0.15	0.20
σ_r MPa	0	7.145	4.939	0

$$\sigma_\theta = \left(\frac{3+\nu}{8}\right)\rho\omega^2\left[r_1^2 + r_2^2 + \frac{r_1^2 r_2^2}{r^2} - \left(\frac{1+3\nu}{3+\nu}\right)r^2\right]$$

$$= \left(\frac{3+0.3}{8}\right)7800\left(\frac{2\pi\times3000}{60}\right)^2 \times \left[0.05^2 + 0.2^2 + \frac{0.05^2\times0.2^2}{r^2} - \frac{1.9}{3.3}\times r^2\right]$$

σ_θ values.

r cm	5	10	15	20
σ_θ MPa	25.748	14.85	10.81	7.014

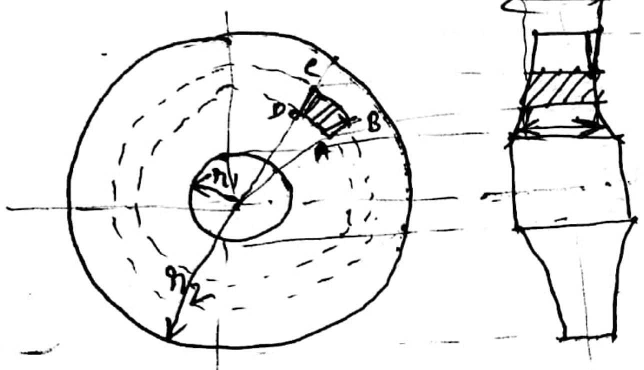


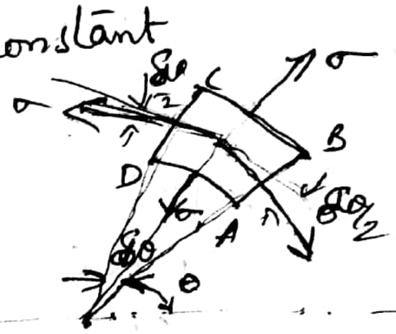
Variation σ_r and σ_θ.

Disc of uniform strength:-

If the disc is to have uniform strength then radial and hoop stresses must be equal all points in the disc.

$\therefore \sigma_r = \sigma_\theta = \sigma = \text{Constant}$





Consider the equilibrium of the Element ABCD

Let Thickness of the disc at radius r is t

Thickness of the disc at radius $(r+\delta r) = t+\delta t$

Now outward radial force acting on face BC

$= \sigma(t+\delta t)(r+\delta r)\,\delta\theta$

$= \sigma(rt + t\delta r + r\delta t)\delta\theta$ (neglect $\delta r \delta t$)

centrifugal force on the element ABCD $= \delta m\, \omega^2 r$

$= \rho(r\,\delta\theta\,\delta r \cdot t) \times \omega^2 r$

Inward radial force acting on face AD $= \sigma \times t \times r\,\delta\theta$.

Inward radial force acting on face AB & CD

$= \sigma \times t\,\delta r \times \delta\theta/2 + \sigma\, t\,\delta r\, \frac{\delta\theta}{2} = \sigma\, t\, \delta r\, \delta\theta$

for equilibrium of the element

Total inward radial force = Total outward radial force

$(\sigma\, t\, r\,\delta\theta) + (\sigma\, t\,\delta r\,\delta\theta) = \sigma(rt + t\delta r + r\delta t)\delta\theta + \rho(r\,\delta\theta\,\delta r\, t)$

$\Rightarrow \sigma\, \eta\, \delta t\, \delta\theta + \rho\, \delta\theta\, \delta\eta\, t\, \omega^2 \eta^2 = 0$

$$\frac{\delta t}{t} = -\rho \frac{\delta\eta}{\eta} \times \eta^2 \times \frac{\omega^2}{\sigma}$$

$$\Rightarrow \quad \frac{dt}{t} = -\rho \frac{\omega^2}{\sigma} \times \eta\, d\eta$$

Integrating the above eq.

$$\ln(t) = -\rho \frac{\omega^2}{\sigma} \times \frac{\eta^2}{2} + \ln A$$

Where $\ln A$ is a constant of integration

$$\therefore \ln\left(\frac{t}{A}\right) = -\rho \left(\frac{\omega^2 \eta^2}{2\sigma}\right)$$

$$\therefore \quad t = A\, e^{-\rho\left(\frac{\omega^2\eta^2}{2\sigma}\right)}$$

Let At $\eta = \eta_1$, $t = t_0$

$$t_0 = A\, e^{-\frac{\rho\omega^2\eta_1^2}{2\sigma}}$$

$$\therefore A = t_0\, e^{\frac{\rho\omega^2\eta_1^2}{2\sigma}}$$

$$\therefore t = t_0\, e^{-\frac{\rho \cdot \omega^2}{2\sigma}(\eta^2 - \eta_1^2)}$$