II/IV B.Tech. (Mech.) B/s.

Subject - Strength of Materials - 2 (18ME401)

Study Material

on 1) Centrifugal Stresses

&

2) Shear Center

Rotating Discs and Cylinders

Learning Objectives

After studying this chapter, you should be able to:

LO 1 Describe the types of stresses in rotating discs and cylinders
LO 2 Determine hoop stress in thin rotating ring
LO 3 Express variation of stresses in solid and hollow rotating discs of uniform thickness
LO 4 Express variation of stresses in solid and hollow long rotating cylinders
LO 5 Deduce expression for variation of thickness of rotating discs of uniform strength
LO 6 Define collapse speed and determine the same for solid and hollow discs

14.1 INTRODUCTION

LO 1

Describe the types of stresses in rotating discs and cylinders

There are machine elements which rotate while performing the required function. These include flywheels, thin rings, circular discs, pulley rims, cylinders and spherical shells, etc. Due to rotation, centrifugal forces act on these elements which give rise to radial, circumferential and longitudinal stresses. In a thin rotating ring, the variation of stresses along the thickness is negligible and can be ignored. In a disc of small axial width, the stress in the axial direction is assumed to be zero. The analysis of the rotating elements will be almost similar to the analysis of thick cylinders. A study of stresses in rotating elements such as rings, discs and cylinders of solid and hollow sections is made in this chapter.

14.2 THIN ROTATING RING

LO 2

Determine hoop stress in thin rotating ring

A ring may be considered thin in the radial direction if the variation of stresses along the thickness is negligible and can be ignored. Consider a thin ring rotating about its centre of mass as shown in Fig. 14.1.

Let ω = angular velocity

r = mean radius

t = thickness of the ring

ρ = density of the material of the ring

Consider an element of the ring subtending an angle $d\theta$ at the centre at an angle θ with the x-axis.

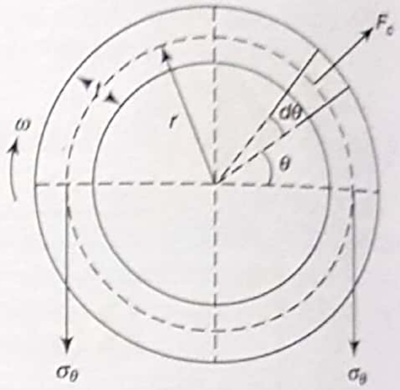


Fig. 14.1

Centrifugal force on the element/unit length

$$= [\rho(r \cdot d\theta)\, t \cdot 1] \cdot r\,\omega^2$$

Vertical component of the force = $\rho r^2 \cdot d\theta \cdot t \cdot \omega^2 \sin\theta$

$$\text{Total vertical force/unit length} = \int_0^\pi \rho \cdot r^2 \cdot d\theta \cdot t \,.\, \omega^2 \sin\theta$$

$$= \rho \cdot r^2 \cdot t \cdot \omega^2 \int_0^\pi \sin\theta \cdot d\theta$$

$$= \rho \cdot r^2 \cdot t \cdot \omega^2 (-\cos\theta)_0^\pi = 2\rho \cdot r^2 \cdot t \cdot \omega^2$$

Let σ_θ = Hoop stress induced in the ring

Then for equilibrium, $\sigma_\theta (2t) \cdot 1 = 2\,\rho \cdot r^2 \cdot t \cdot \omega^2$

$$\sigma_\theta = \rho \cdot r^2\,\omega^2 = \rho \cdot v^2 \tag{14.1}$$

where v is the mean tangential velocity of the ring.

Example 14.1 The rim of a rotating wheel is 1.2 m in diameter. Determine the limiting speed of the wheel and the change in diameter if the maximum stress is not to exceed 130 MPa. Density of the material is 7700 kg/m³ and E = 205 GPa. Neglect the effect of spokes of the wheel. Treat the rim to be thin.

Solution

Given Rim of a rotating wheel

d = 1.2 m $\qquad \sigma_\theta = 130\ \text{N/mm}^2 = 130 \times 10^6\ \text{N/m}^2$

$\rho = 7700\ \text{kg/m}^3 \qquad E = 205$ GPa

To find

— Limiting speed of wheel

— Change in diameter

$$r = 1.2/2 = 0.6\ \text{m}$$

Limiting speed

$$\sigma_\theta = \rho \cdot r^2\,\omega^2$$

or $$130 \times 10^6 = 7700 \times 0.6^2 \times \omega^2$$

or $$\omega = 216.6\ \text{rad/s} \quad \text{or} \quad N = \frac{216.6 \times 60}{2\pi} = 2068\ \text{rpm}$$

Change in diameter

Hoop strain, $$\varepsilon = \frac{\delta d}{d} = \frac{\sigma_\theta}{E}$$

or $$\delta d = \frac{\sigma_\theta}{E} \cdot d = \frac{130}{205\,000} \times 1200 = 0.76\ \text{mm}$$

(Note the consistency of units in the two relations used)

Example 14.2 A flywheel with a moment of inertia of 300 kg · m² rotates at 300 rpm. If the maximum stress is not to exceed 6 MPa, find the thickness of the rim. Take the width of the rim as 150 mm and the density of the material 7400 kg/m³. Neglect the effect of inertia of spokes.

Solution

Given A flywheel rim

$I = 300$ kg · m² $\qquad \sigma_\theta = 6$ N/mm² $= 6 \times 10^6$ N/m²

$\rho = 7400$ kg/m³ $\qquad w = 0.15$ m

$N = 300$ rpm

To find Thickness of rim

$$\omega = \frac{2\pi \times 300}{60} = 10\pi \text{ rad/s}$$

Determination of outer radius

The stress is maximum at the outer radius r_o,

$$\sigma_\theta = \rho \cdot r_o^2 \omega^2$$

or $$6 \times 10^6 = 7400\, r_o^2 \times (10\pi)^2$$

or $$r_o^2 = 0.82 \quad \text{or} \quad r_o = 0.9 \text{ m}$$

Determination of thickness

As an approximation, initially assume some value of the mean radius and the radius of gyration. Let it be 0.85 m (a little less than 0.9 m). Then if t is the thickness,

$$\text{Moment of inertia} = [(2\pi r \cdot w \cdot t)\, \rho]\, r^2$$

or $$300 = (2\pi \times 0.85 \times 0.15 \times t \times 7400) \times 0.85^2$$

or $$t = 0.07 \text{ m}$$

Inner radius, $r_i = 0.9 - 0.07 = 0.83$ m and mean radius $= \dfrac{0.9 + 0.83}{2} = 0.865$ m

If k is the radius of gyration, $k^2 = \dfrac{0.9^2 + 0.83^2}{2} = 0.749$ m²

∴ moment of inertia, $I = mk^2$ or $300 = (2\pi \times 0.865 \times 0.15 \times t \times 7400) \times 0.749$

or $t = 0.066$ m or 66 mm

which approximately satisfies the assumption of mean radius (865 mm) and outer radius (900 mm).

Example 14.3 A built-up ring is made up of two materials. The outer ring is of steel and the inner one of copper. The diameter of the common surfaces is 800 mm. Each ring has a width of 30 mm and a thickness of 20 mm in the radial direction. The ring rotates at 1800 rpm. Find the stresses set up in the steel and the copper. $E_s = 2E_c$; Density of steel = 7300 kg/m³; Density of copper = 9000 kg/m³.

Solution

Given A built-up ring is made up of two materials

$d = 800$ mm $\qquad t = 20$ mm

$w = 30$ mm $\qquad N = 1800$ rpm

$\rho_s = 7300$ kg/m³ $\qquad \rho_c = 9000$ kg/m³

$E_s = 2E_c$

To find Stresses in steel and copper

Refer Fig. 14.2.

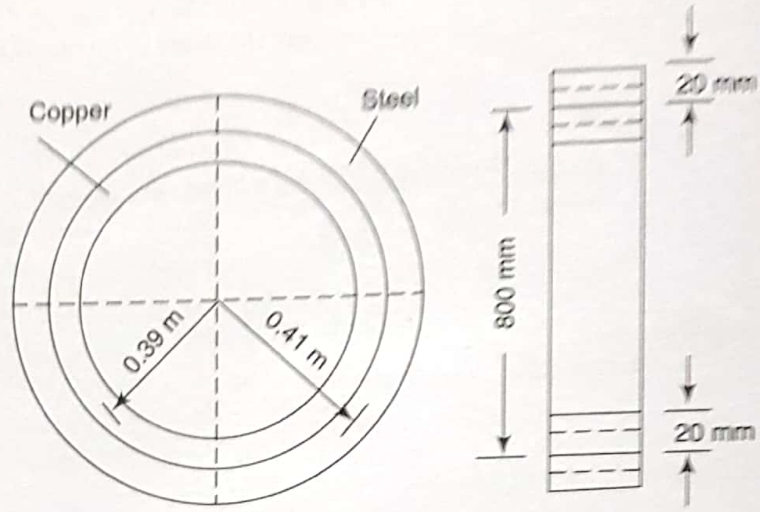


Fig. 14.2

$$r = 400 \text{ mm} = 0.4 \text{ m}; \ \omega = \frac{2\pi \times 1800}{60} = 60\pi \text{ rad/s}$$

Let p be the shrinkage pressure at the common surface at stand still.

Hoop stress due to shrinkage

$$\text{In the steel ring} = \frac{pd}{2t} = \frac{p \times 800}{2 \times 20} = 20p \qquad \text{(tensile)}$$

(as in case of thin cylinder with internal pressure)

$$\text{In the copper ring} = \frac{p \times 800}{2 \times 20} = 20p \qquad \text{(compressive)}$$

(as in case of thin cylinder with external pressure)

Hoop stress due to rotation

In steel, $\sigma_\theta = \rho \cdot r^2 \omega^2 = 7300(0.4 + 0.01)^2 \times (60\pi)^2$

$= 43.6 \times 10^6 \text{ N/m}^2$ or 43.6 MPa (tensile)

In copper, $\sigma_\theta = 9000(0.4 - 0.01)^2 \times (60\pi)^2$

$= 48.6 \times 10^6 \text{ N/m}^2$ or 48.6 MPa (tensile)

Equating the net strains

As the net strains of the two must be equal,

$$\frac{20p + 43.6}{E_s} = \frac{-20p + 48.6}{E_c}$$

or

$$\frac{20p + 43.6}{2E_c} = \frac{-20p + 48.6}{E_c}$$

or $20p + 43.6 = 2(48.6 - 20p)$ or $p = 0.893$ MPa

Total stress in steel $= 20 \times 0.893 + 43.6 = 61.5$ MPa

Total stress in copper $= -20 \times 0.893 + 48.6 = 30.7$ MPa

14.3 DISC OF UNIFORM THICKNESS

LO 3

Express variation of stresses in solid and hollow rotating discs of uniform thickness

Consider a flat rotating disc of uniform thickness t and having R_o and R_i as the outer and the inner radii respectively. Let the angular speed of the disc be ω.

For a disc of small axial width, it may be assumed that the stress in the axial direction is zero.

For an element of the disc of unit thickness (Fig. 14.3), let

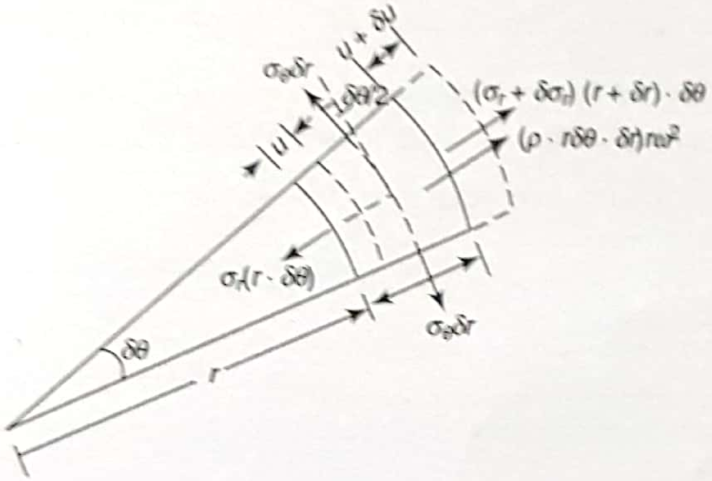


Fig. 14.3

σ_r = radial stress at the inner face at radius r of the element

$\sigma_r + \delta\sigma_r$ = radial stress at the outer face at radius $(r + \delta r)$ of the element

σ_θ = circumferential (hoop) stress on the radial faces

As the disc rotates, let u be the radial shift at an unstrained radius r, i.e. r becomes $r+u$ after straining. Similarly, $u+\delta u$ be the radial shift at an unstrained radius $r + \delta r$.

- Radial strain $= \dfrac{\text{Increase in } \delta r}{\delta r} = \dfrac{u + \delta u - u}{dr} = \dfrac{du}{dr}$ in the limit

$$\therefore \text{ radial stress} = E \cdot \frac{du}{dr} = \sigma_r - \nu\sigma_\theta \quad (14.2)$$

- Circumferential (hoop) strain $= \dfrac{\text{Increase in circumference}}{\text{original circumference}} = \dfrac{2\pi(r+u) - 2\pi r}{2\pi r} = \dfrac{u}{r}$

$$\therefore \text{ circumferential stress} = E \cdot \frac{u}{r} = \sigma_\theta - \nu\sigma_r$$

or $$E \cdot u = r(\sigma_\theta - \nu\sigma_r) \quad (14.3)$$

Differentiating it, $$E \cdot \frac{du}{dr} = r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) + \sigma_\theta - \nu\sigma_r$$

From Eqs. 14.2 and 14.3,

$$\sigma_r - \nu\sigma_\theta = r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) + \sigma_\theta - \nu\sigma_r$$

or $$(1+\nu)(\sigma_r - \sigma_\theta) = r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right)$$

or $$\sigma_r - \sigma_\theta = \frac{r}{1+\nu}\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) \tag{14.4}$$

- For equilibrium of forces in radial direction, considering a unit length of element,

Centrifugal force $= mr\omega^2 = [\rho(r\,\delta\,\theta)\cdot\delta r]\,r\,\omega^2$ (outward)

Radial force on inner face $= \sigma_r\,(r\cdot\delta\theta)$ (inward)

Radial force on outer face $= (\sigma_r + \delta\sigma_r)(r+\delta r)\cdot\delta\theta$ (outward)

Radial components of tangential force $= 2\sigma_\theta\cdot(\delta r\cdot 1)\sin\frac{1}{2}\delta\theta \approx \sigma_\theta\cdot\delta r\cdot\delta\theta$ (inward)

For equilibrium, Net inward force = Net outward force

$$\sigma_\theta\cdot\delta r\cdot\delta\theta + \sigma_r r\cdot\delta\theta - (\sigma_r+\delta\sigma_r)(r+\delta r)\cdot\delta\theta = \rho(r\delta\theta)\cdot\delta r\cdot r\omega^2$$

or $$\sigma_\theta\cdot\delta r + \sigma_r r - (\sigma_r+\delta\sigma_r)(r+\delta r) = \rho r^2\cdot\delta r\cdot\omega^2$$

Simplifying and taking limits,

$$\sigma_\theta\cdot dr + \sigma_r r - (\sigma_r r + \sigma_r dr + r\cdot d\sigma_r) = \rho r^2\omega^2\cdot dr$$

$$\sigma_\theta\cdot dr - \sigma_r dr - r\cdot d\sigma_r) = \rho r^2\omega^2\cdot dr$$

Dividing by *dr*, the equilibrium equation is obtained,

$$\sigma_\theta - \sigma_r - \frac{r\cdot d\sigma_r}{dr} = \rho r^2\omega^2$$

or $$\sigma_r - \sigma_\theta = -\frac{r\cdot d\sigma_r}{dr} - \rho r^2\omega^2 \tag{14.5}$$

From Eqs. 14.4 and 14.5,

$$\frac{r}{1+\nu}\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) = -\frac{r\cdot d\sigma_r}{dr} - \rho r^2\omega^2$$

or $$\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr} + (1+\nu)\frac{d\sigma_r}{dr} = -\rho r^2\omega^2(1+\nu)$$

or $$\frac{d\sigma_\theta}{dr} - \frac{d\sigma_r}{dr}(\nu - 1 - \nu) = -\rho r\omega^2(1+\nu)$$

or $$\frac{d}{dr}(\sigma_\theta + \sigma_r) = -(1+\nu)\rho r\omega^2$$

Integrating, $$\sigma_r + \sigma_\theta = -\frac{1}{2}(1+\nu)\rho r^2\omega^2 + 2A \tag{14.6}$$

Adding Eqs. 14.5 and 14.6,

$$2\sigma_r + r\cdot\frac{d\sigma_r}{dr} + \rho r^2\omega^2 = -\frac{1}{2}(1+\nu)\rho r^2\omega^2 + 2A$$

or $$\left(2\sigma_r + r\cdot\frac{d\sigma_r}{dr}\right) = -\frac{3+\nu}{2}\rho r^2\omega^2 + 2A$$

or $$\frac{1}{r}\cdot\frac{d(r^2\cdot\sigma_r)}{dr} = -\frac{3+\nu}{2}\rho r^2\omega^2 + 2A$$

or $$\frac{d(r^2 \cdot \sigma_r)}{dr} = -\frac{3+\nu}{2}\rho r^3\omega^2 + 2Ar$$

Integrating, $$r^2 \cdot \sigma_r = -\frac{3+\nu}{8}\rho r^4\omega^2 + Ar^2 - B$$

or $$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho r^2\omega^2 \quad (14.7)$$

Inserting this value in Eq. 14.6,

$$A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho r^2\omega^2 + \sigma_\theta = -\frac{1}{2}(1+\nu)\rho r^2\omega^2 + 2A$$

$$\sigma_\theta = -\frac{1}{2}(1+\nu)\rho r^2\omega^2 + A + \frac{B}{r^2} + \frac{3+\nu}{8}\rho r^2\omega^2 = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho r^2\omega^2 \quad (14.8)$$

Equations 14.7 and 14.8 are the governing equations of a rotating disc of uniform thickness.

Solid Disc

In a solid disc, the stress at the centre, $\sigma_r = A - \frac{B}{0}$, i.e., infinite which is not possible,

∴ B has to be zero.

Let R be the outside diameter. Then at the outer surface, $\sigma_R = 0$,

Therefore $0 = A - \frac{3+\nu}{8}\rho R^2\omega^2$ or $A = \frac{3+\nu}{8}\rho R^2\omega^2$

Thus $$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R^2 - r^2) \quad (14.9)$$

and $$\sigma_\theta = \frac{\rho\omega^2}{8}[(3+\nu)R^2 - (1+3\nu)r^2] \quad (14.10)$$

- At the centre, $r = 0$, thus $\sigma_r = \sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 R^2$ (maximum value of stress) (14.11)
- At the outer surface, $\sigma_\theta = \frac{1-\nu}{4}\rho\omega^2 R^2$ and $\sigma_r = 0$ (14.12)
- If $\nu = 0.3$, $\sigma_r = \sigma_\theta = \frac{3+0.3}{8}\rho\omega^2 R^2 = 0.413\rho\omega^2 R^2$ at the centre (maximum)

and at the outer surface, $\sigma_\theta = \frac{1-0.3}{4}\rho\omega^2 R^2 = 0.175\rho\omega^2 R^2$

$$= \frac{0.175}{0.413}\sigma_{\theta(\max)} = 0.424\sigma_{\theta(\max)}$$

The variation of stresses with radius is shown in Fig. 14.4.

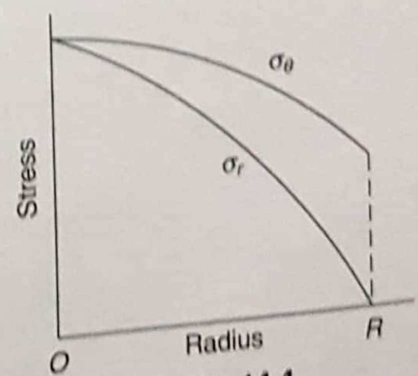


Fig. 14.4

Hollow Disc

Let R_i and R_o be the inside and the outside radii of a hollow disc respectively. Radial stresses are to be zero at these values.

$$\sigma_r = 0 = A - \frac{B}{R_i^2} - \frac{3+\nu}{8}\rho R_i^2\omega^2 \tag{i}$$

$$\sigma_r = 0 = A - \frac{B}{R_o^2} - \frac{3+\nu}{8}\rho R_o^2\omega^2 \tag{ii}$$

From (i) from (ii), $-\frac{B}{R_i^2} - \frac{3+\nu}{8}\rho R_i^2\omega^2 = -\frac{B}{R_o^2} - \frac{3+\nu}{8}\rho R_o^2\omega$

Multiplying throughout by $R_i^2 R_o^2$,

or $$-BR_o^2 - \frac{3+\nu}{8}\rho R_i^4 R_o^2\omega^2 + BR_i^2 + \frac{3+\nu}{8}\rho R_i^2 R_o^4\omega = 0$$

$$B(R_i^2 - R_o^2) - \frac{3+\nu}{8}\rho R_i^2 R_o^2\omega^2(R_i^2 - R_o^2) = 0$$

or $$B = \frac{3+\nu}{8}\rho R_i^2 R_o^2\omega^2$$

and thus $\sigma_r = 0 = A - \frac{3+\nu}{8}\rho R_o^2\omega^2 - \frac{3+\nu}{8}\rho R_i^2\omega^2$ [from (i)]

or $$A = \frac{3+\nu}{8}\rho\omega^2(R_i^2 + R_o^2)$$

Inserting the values of constants A and B in Eq. 14.7,

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R_i^2 + R_o^2) - \frac{3+\nu}{8r^2}\rho R_i^2 R_o^2\omega^2 - \frac{3+\nu}{8}\rho r^2\omega^2$$

$$= \frac{\rho\omega^2}{8}(3+\nu)\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{r^2} - r^2\right) \tag{14.13}$$

Inserting the values of constants A and B in Eq. 14.8,

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{r^2}\right) - (1+3\nu)r^2\right] \tag{14.14}$$

- σ_r is maximum when $\frac{d\sigma_r}{dr} = 0$ or $\frac{2R_i^2 R_o^2}{r^3} - 2r = 0$

or $$R_i^2 R_o^2 - r^4 = 0 \text{ or at } r = \sqrt{R_i R_o} \tag{14.15}$$

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{R_i R_o} - R_i R_o\right)$$

$$= \frac{3+\nu}{8}\rho\omega^2(R_i^2 + R_o^2 - 2R_i R_o) = \frac{3+\nu}{8}\rho\omega^2(R_o - R_i)^2 \tag{14.16}$$

- σ_θ is maximum at inside ($r = R_i$),

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{R_i^2}\right) - (1+3\nu)R_i^2\right]$$

$$= \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \qquad (14.17)$$

- If R_i is very small, $\sigma_\theta = \frac{\rho\omega^2}{4}[(3+\nu)R_o^2]$, i.e., twice that for a solid disc. (14.18)
- At outside, $\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{R_o^2}\right) - (1+3\nu)R_o^2\right]$

$$= \frac{\rho\omega^2}{8}[(3+\nu)(2R_i^2 + R_o^2) - (1+3\nu)R_o^2]$$

$$= \frac{\rho\omega^2}{8}[(3+\nu)2R_i^2 + (3+\nu)R_o^2 - (1+3\nu)R_o^2]$$

$$= \frac{\rho\omega^2}{4}[(3+\nu)R_i^2 + (1-\nu)R_o^2] \qquad (14.19)$$

- In a thin rotating ring, $R_i \to R_o = R$

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R^2 + R^2 + \frac{R^2R^2}{R^2}\right) - (1+3\nu)R^2\right] = \rho\omega^2R^2$$

the same result as obtained in Section 14.2.

The variation of stresses is shown in Fig. 14.5.

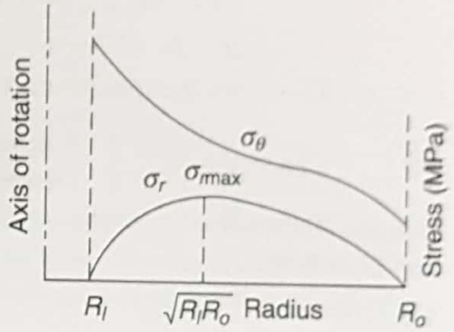


Fig. 14.5

Example 14.4 A disc of uniform thickness and of 600-mm diameter rotates at 1800 rpm. Find the maximum stress developed in the disc. If a hole of 100-mm diameter is made at the centre of the disc, find the maximum values of radial and hoop stresses. Density of the material of the disc = 7700 kg/m³ and ν = 0.3.

Solution

Given A solid disc of uniform thickness

$d = 600$ mm $\qquad N = 1800$ rpm

$\rho = 7700$ kg/m³ $\qquad \nu = 0.3$

To find

- Maximum stress
- Maximum radial and hoop stresses when a hole of 100 mm is made

$$R = 0.3 \text{ m}; \qquad \omega = \frac{2\pi \times 1800}{60} = 60\pi \text{ rad/s}$$

For solid disc

Maximum radial stress and hoop stress are at the centre and are equal,

$$\sigma_r = \sigma_\theta = \frac{3+\nu}{8}\rho\omega^2R^2 = \frac{3+0.3}{8} \times 7700 \times (60\pi)^2 \times 0.3^2 \qquad \text{...(Eq. 14.11)}$$

$$= 112.85 \times 10^6 \times 0.3^2 = 10.16 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad 10.16 \text{ MPa}$$

When a hole is made

$R_i = 0.05$ m and $R_o = 0.3$ m

Maximum radial stress is at $\sqrt{R_iR_o}$ radius, i.e., at radius $\sqrt{50 \times 300} = 122.5$ mm

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R_o - R_i)^2 \quad \text{...(Eq. 14.15)}$$

$$= \frac{3+0.3}{8} \times 7700 \times (60\pi)^2(0.3-0.05)^2 \quad \text{...(Eq. 14.16)}$$

$$= 122.85 \times 10^6 \times 0.0625 = 7.05 \times 10^6 \text{ N/m}^2 \text{ or } 7.05 \text{ MPa}$$

Maximum hoop stress is at the inner radius,

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \quad \text{...(Eq. 14.17)}$$

$$= \frac{7700 \times (60\pi)^2}{4}[(1-0.3)\times 0.05^2 + (3+0.3)\times 0.3^2]$$

$$= 68.396 \times 10^6 \times 0.298\,75 = 20.43 \times 10^6 \text{ N/m}^2$$

or 20.43 MPa

Figure 14.6 shows the maximum values of hoop and radial stresses.

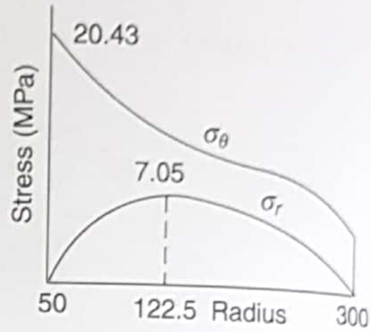


Fig. 14.6

Example 14.5 A solid disc of uniform thickness and having a diameter of 400 mm rotates at 7500 rpm. Determine the radial and the hoop stresses at radii of 0, 50 mm, 100 mm, 150 mm and 200 mm. Density of the material is 7500 kg/m². What are the maximum values of the radial, hoop and shear stresses?

Solution

Given A solid disc of uniform thickness

$d = 400$ mm $\quad N = 7500$ rpm

$\rho = 7500$ kg/m³

To find

— Radial and hoop stresses at 0, 50 mm, 100 mm, 150 mm and 200 mm

— maximum values of radial, hoop and shear stresses

$$R = 200 \text{ mm} = 0.2 \text{ m};\ \omega = \frac{2\pi \times 7500}{60} = 250\pi \text{ rad/s}$$

Radial stresses

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R^2 - r^2) \quad \text{...(Eq. 14.9)}$$

$$= \frac{3+0.25}{8} \times 7500 \times (250\pi)^2(0.2^2 - r^2)$$

$$= 1879.5 \times 10^6 (0.04 - r^2) \text{ N/m}^2$$

$$= 1879.5(0.04 - r^2) \text{ MPa}$$

R (m)	0	0.05	0.1	0.15	0.2
σ_r (MPa)	75.2	70.5	56.4	32.9	0

Hoop stresses

and $$\sigma_\theta = \frac{\rho\omega^2}{8}[(3+\nu)R^2 - (1+3\nu)r^2] \quad \text{...(Eq. 14.10)}$$

$$= \frac{7500 \times (250\pi)^2}{8}[(3+0.25)\times 0.2^2 - (1+3\times 0.25)r^2]$$
$$= 578.3 \times 10^6\,(0.13 - 1.75r^2)\ \text{N/m}^2$$
$$= 578.3(0.13 - 1.75r^2)\ \text{MPa}$$

or

R (m)	0	0.05	0.1	0.15	0.2
σ_θ (MPa)	75.2	72.6	65.1	52.4	34.7

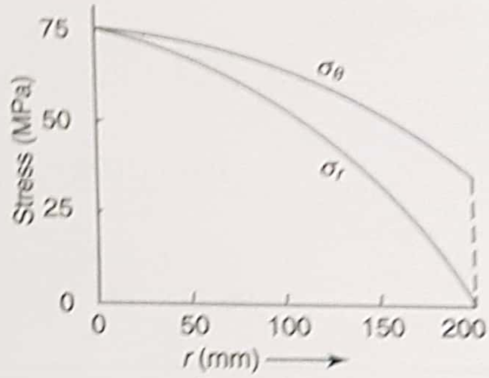


Fig. 14.7

Maximum stresses

Maximum radial stress = maximum hoop stress = 75.2 MPa

The principal stresses at any point are σ_r, σ_θ and zero (along axial direction).

$$\therefore \text{ maximum shear stress } = \frac{75.2 - 0}{2} = 37.6 \text{ MPa}$$

The variation of stresses is shown in Fig. 14.7.

Example 14.6 A thin disc of uniform thickness is of 800-mm outer diameter and 50-mm inner diameter. It rotates at 3000 rpm. Determine the radial and the hoop stresses at radii of 0, 25 mm, 50 mm, 100 mm, 150 mm, 200 mm, 300 mm and 400 mm. Density of the material is 7800 kg/m². $\nu = 0.25$. What are the maximum values of the radial, hoop and shear stresses?

Solution

Given A thin disc of uniform thickness

$R_i = 0.025$ m $\quad R_o = 0.4$ m $\quad N = 3000$ rpm

$\rho = 7800$ kg/m³ $\quad \nu = 0.25$

To find

- Radial and hoop stresses at 0, 25 mm, 50 mm, 100 mm, 150 mm, 200 mm, 300 mm and 400 mm
- Maximum values of radial, hoop and shear stresses

$$\omega = \frac{2\pi \times 3000}{60} = 100\pi \text{ rad/s}$$

Radial stresses

$$\sigma_r = \frac{\rho\omega^2}{8}(3+\nu)\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{r^2} - r^2\right) \qquad \text{...(Eq. 14.13)}$$

$$= \frac{7800 \times (100\pi)^2}{8}(3+0.25)\left(0.025^2 + 0.4^2 - \frac{0.025^2 \times 0.4^2}{r^2} - r^2\right)$$

$$= 312.750 \times 10^6 \left(0.1606 - \frac{0.0001}{r^2} - r^2\right) \text{N/m}^2$$

$$= 312.75\left(0.1606 - \frac{0.0001}{r^2} - r^2\right) \text{MPa}$$

r (m)	0.025	0.05	0.1	0.15	0.2	0.3	0.4
σ_r (MPa)	0	36.94	43.97	41.8	36.94	21.73	0

Hoop stresses

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{r^2}\right) - (1+3\nu)r^2\right] \quad \text{...(Eq. 14.14)}$$

$$= \frac{7800\times(100\pi)^2}{8}\left[(3+0.25)\left(0.025^2+0.4^2+\frac{0.025^2\times 0.4^2}{r^2}\right)-(1+3\times 0.25)r^2\right]$$

$$= 96.23\times 10^6\left[3.25\left(0.1606+\frac{0.0001}{r^2}\right)-1.75r^2\right]$$

R (m)	0.025	0.05	0.1	0.15	0.2	0.3	0.4
σ_r (MPa)	100.17	62.32	51.68	47.83	44.28	35.42	23.48

Maximum stresses

Maximum radial stress is at radius $\sqrt{R_i R_o} = \sqrt{25\times 400} = 100$ mm and is 43.97 MPa.

Maximum hoop stress = 100.17 MPa

The principal stresses at the inner surface are 100.17 MPa, 0 and 0 (along axial direction)

$$\therefore \quad \text{maximum shear stress} = \frac{100.17-0}{2} = 50.09 \text{ MPa}$$

The variation of stresses is shown in Fig. 14.8.

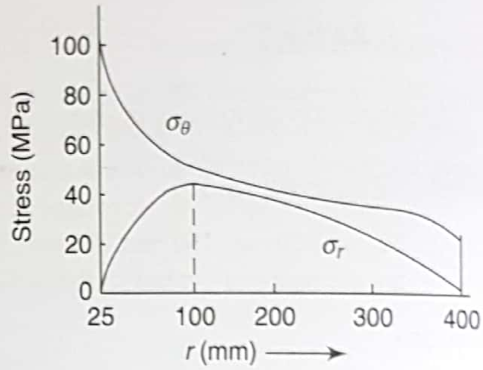


Fig. 14.8

Example 14.7 A hollow steel disc of 400-mm outer diameter and 100-mm inside diameter is shrunk on a steel shaft. The pressure between the disc and the shaft is 60 MPa. Determine the speed of the disc at which it will loosen from the shaft neglecting the change in the dimensions of the shaft due to rotation. ρ = 7700 kg/m³ and ν = 0.3.

Solution

Given A hollow steel disc shrunk on a solid steel shaft

$R_i = 0.05$ m $\qquad R_o = 0.2$ m

$p = 60$ MPa $\qquad \rho = 7700$ kg/m³

$\nu = 0.3$

To find Speed of disc at which it will loosen from the shaft

At stand still

At stand still, the hollow disc on a shaft is similar to a hub on a solid shaft (Section 13.8) and thus the hollow steel disc is equivalent to a thick cylinder subjected to an internal pressure and thus the results of the same may be used.

As the change in the dimensions of the shaft due to rotation is to be neglected, the change in the hollow steel disc is only to be considered.

Maximum stress is at the inner radius and is given by Eq. 13.20.

$$\sigma_\theta = \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2}\cdot p_i = \frac{0.2^2+0.05^2}{0.2^2-0.05^2}\times 60 = 68 \text{ MPa}$$

$$\text{Hoop strain at the inner radius} = \frac{\sigma_\theta - \nu\sigma_r}{E} = \frac{68+0.3\times 60}{E} = \frac{86}{E}$$

And the shrink allowance = $\frac{86}{E} \times 50$

The shrink allowance is the difference of diameters before shrinking.

When the disc rotates

Maximum hoop stress is at the inner radius,

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \qquad \text{...(Eq. 14.17)}$$

$$= \frac{7700 \times \omega^2}{4}[(1-0.3) \times 0.05^2 + (3+0.3) \times 0.2^2]$$

$$= 1925\omega^2\,(0.133\,75) = 257.47\omega^2 \text{ N/m}^2 \quad \text{or} \quad 257.47 \times 10^{-6}\,\omega^2 \text{ N/mm}^2$$

Radial stress is zero at the inner radius when the disc loosens.

Hoop strain at the inner radius = $\frac{\sigma_\theta - \nu\sigma_r}{E} = \frac{\sigma_\theta - 0}{E} = \frac{257.47 \times 10^{-6}\omega^2}{E}$

Increase in inner radial distance at speed ω for the disc = $\frac{257.47 \times 10^{-6}\omega^2}{E} \times 50$

If disc is to be loosen

If the disc is to become loosen, the increase in inner radial distance at speed ω for the disc must be equal to the shrink allowance,

i.e.,
$$\frac{257.47 \times 10^{-6}\omega^2}{E} \times 50 = \frac{86}{E} \times 50$$

or
$$\omega = 577.9 \text{ rad/s} \quad \text{or} \quad N = \frac{577.9 \times 60}{2\pi} = 5519 \text{ rpm}$$

Example 14.8 A hollow steel disc of 600-mm outer diameter and 200-mm inside diameter is shrunk on a solid steel shaft. The shrinkage is 1 in 1000. Determine

(i) the stresses at stand still
(ii) the speed at which shrink fit will loosen
(iii) the maximum stress in the disc at that speed
(iv) the hoop stress in the disc at half the speed found in (ii)

$\rho = 7600$ kg/m³ and $\nu = 0.3$, $E = 205$ GPa

Solution

Given A hollow steel disc shrunk on a solid steel shaft

$R_i = 0.1$ m	$R_o = 0.3$ m
$\rho = 7600$ kg/m³	$\nu = 0.3$
$\delta/R = 1/1000$	$E = 205$ GPa

To find

— stresses at stand still
— speed at which shrink fit loosen
— maximum stress at that speed
— hoop stress at half speed

At stand still

Let the shrinkage pressure between the disc and the shaft at stand still be p.

$$\text{Shrink allowance} = 100 \times \frac{1}{1000} = 0.1 \text{ mm}$$

At stand still, the hollow disc acts similar to a thick cylinder subjected to internal pressure (Refer Section 13.9). If the change in dimensions of both, the shaft and the hollow disc are to be considered, the derived expression of Eq. 13.28 can be used directly.

$$\text{Shrinkage allowance (initial difference in radii)} = \frac{2pR_iR_o^2}{E(R_o^2 - R_i^2)} \qquad \text{...(Eq. 13.28)}$$

or
$$0.1 = \frac{2p \times 100 \times 300^2}{205\,000(300^2 - 100^2)}$$

or
$$p = 91.1 \text{ MPa}$$

Maximum hoop stress is at the inner radius and is given by Eq. 13.20.

$$\sigma_\theta = \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} \cdot p_i$$

$$= \frac{0.3^2 + 0.1^2}{0.3^2 - 0.1^2} \cdot p = 1.25p = 1.25 \times 91.1 = 113.9 \text{ MPa}$$

When shrink fit loosen

When the disc rotates and the shrink fit loosens, radial pressure is zero and thus the radial stress is also zero.

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \qquad \text{...(Eq. 14.17)}$$

$$= \frac{7600 \times \omega^2}{4}\left[(1-0.3) \times 0.1^2 + (3+0.3) \times 0.3^2\right]$$

$$= 577.6\omega^2 \text{ N/m}^2 \quad \text{or} \quad 577.6 \times 10^{-6}\omega^2 \text{ N/mm}^2$$

and for the solid shaft,

$$\sigma_\theta = \frac{1-\nu}{4}\rho\omega^2R^2 \qquad \text{...(Eq. 14.12)}$$

$$= \frac{1-0.3}{4} \times 7600 \times \omega^2 \times 0.1^2 = 13.3\omega^2 \text{ N/m}^2 \text{ or } 13.3 \times 10^{-6}\,\omega^2 \text{ N/mm}^2$$

As radial strains are equal to hoop strains,

Increase in inner radial distance at speed ω for the solid shaft $= \dfrac{13.3 \times 10^{-6}\omega^2}{E} \times 100$

However, initially the radius of the solid shaft is more than the inner radius of disc by 0.1 mm. Therefore, the inner radius of the disc must increase by 0.1 mm more relative to that of the solid shaft. Thus

$$\frac{577.6 \times 10^{-6}\omega^2}{E} \times 100 = \frac{13.3 \times 10^{-6}\omega^2}{E} \times 100 + 0.1$$

$$\frac{(577.6-13.3)\times10^{-6}\omega^2}{205\,000}\times100=0.1$$

or
$$\omega^2=363\,282$$

or
$$\omega=602.7\text{ rad/s}\quad\text{or}\quad N=\frac{602.7\times60}{2\pi}=5756\text{ rpm}$$

Maximum stresses

$$\sigma_\theta=577.6\omega^2=577.6\times363\,282=209.8\times10^6\text{ N/m}^2$$

$$\sigma_r=\frac{3+\nu}{8}\rho\omega^2(R_o-R_i)^2 \qquad \text{...(Eq. 14.16)}$$

$$=\frac{3+0.3}{8}\times7600\times363\,282(0.3-0.1)^2$$

$$=45.56\times10^6\text{ N/m}^2\quad\text{or}\quad45.56\text{ MPa}$$

Hoop stress at half speed

The hoop stress in the disc varies as square of speed.

Hoop stress at zero speed = 113.9 MPa

Hoop stress at 602.7 rad/s = 209.8 MPa

∴ hoop stress in the disc at half the speed

$$=113.9+(209.8-113.9)\cdot\left(\frac{1}{2}\right)^2=137.9\text{ MPa}$$

Example 14.9 A hollow steel disc of 300-mm outer diameter and 100-mm inside diameter is shrunk on a hollow cast iron disc of 40-mm internal diameter. Determine the change in the shrink fit pressure when the assembly rotates at 4800 rpm.

ρ_s = 7700 kg/m³ ; ρ_{ci} = 7000 kg/m³ and ν = 0.3 for both, $E_s = 2E_{ci}$

Solution

Given A hollow steel disc on a hollow cast-iron disc

Steel disc: $R_i = 0.05$ m $\quad R_o = 0.15$ m

Cast iron disc: $R_i = 0.02$ m $\quad R_o = 0.05$ m

$\rho_s = 7700$ kg/m³ $\quad \rho_{ci} = 7000$ kg/m³

$\nu = 0.3$ $\quad E_s = 2E_{ci}$

$N = 4800$ rpm

To find Change in shrink fit pressure

$$\omega=\frac{2\pi\times4800}{60}=160\pi$$

At stand still

Let the shrinkage pressure between the disc and the shaft at stand still be p.

At stand still, the hollow disc acts similar to a thick cylinder subjected to internal pressure only and thus the results of the same may be used.

- For the outer disc, hoop stress at the inner radius (50 mm),

$$R_i=0.05\text{ m}\quad\text{and}\quad R_o=0.15\text{ m}$$

$$\sigma_\theta = \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} \cdot p_i = \frac{0.15^2 + 0.05^2}{0.15^2 - 0.05^2} \cdot p = 1.25p \text{ MPa}$$

$$\text{Hoop strain} = \frac{1.25p + 0.3p}{E_s} = \frac{1.55p}{E_s}$$

- For the inner disc, hoop stress is at the outer radius (50 mm),
  $R_i = 0.02$ m and $R_o = 0.05$ m

$$\sigma_\theta = -\frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} p_o = -\frac{0.05^2 + 0.02^2}{0.05^2 - 0.02^2} \cdot p = -1.381p \text{ MPa}$$

$$\text{Hoop strain} = \frac{-1.381p + 0.3p}{E_{ci}} = \frac{-1.081p}{E_{ci}}$$

When rotating at 4800 rpm

Let the shrinkage pressure between the disc and the shaft be p'.

For the outer disc, hoop stress at 50 mm radius = $1.25\,p'$ (as above)

For the inner disc, hoop stress at 50 mm radius = $-1.381p'$ (as above)

Due to rotation,

For outer disc at 50-mm radius, $\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2]$ ...(Eq. 14.17)

$$= \frac{7700 \times (160\pi)^2}{4}[(1-0.3)\times 0.05^2 + (3+0.3)\times 0.15^2]$$

$$= 36.964 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad 36.964 \text{ MPa}$$

For the inner disc at 50-mm radius,

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(3+\nu)R_i^2 + (1-\nu)R_o^2] \qquad \text{...(Eq. 14.19)}$$

$$\sigma_\theta = \frac{7000 \times (160\pi)^2}{4}[(3+0.3)\times 0.02^2 + (1-0.3)\times 0.05^2]$$

$$= 1.357 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad 1.357 \text{ MPa}$$

$$\text{Hoop strain in outer disc} = \frac{(1.25p' + 36.964\times 10^6) + 0.3p'}{E_s} = \frac{1.55p' + 36.964\times 10^6}{E_s}$$

$$\text{Hoop strain in inner disc} = \frac{(-1.381p' + 1.357\times 10^6) + 0.3p'}{E_{ci}} = \frac{-1.081p' + 1.357\times 10^6}{E_{ci}}$$

For equilibrium

$$\frac{1.55p}{E_s} - \frac{-1.081p}{E_{ci}} = \frac{1.55p' + 36.964\times 10^6}{E_s} - \frac{-1.081p' + 1.357\times 10^6}{E_{ci}}$$

or $$\frac{1.55p}{2E_{ci}} - \frac{-1.081p}{E_{ci}} = \frac{1.55p' + 36.964\times 10^6}{2E_{ci}} - \frac{-1.081p' + 1.357\times 10^6}{E_{ci}}$$

or $$1.55p + 2(1.081p) = 1.55p' + 36.964 + 2(1.081p' - 1.357)$$

or $$3.712(p - p') = 34.25$$

or $$p - p' = 9.22 \text{ MPa}$$

LO 5

Deduce expression for variation of thickness of rotating discs of uniform strength

14.5 DISC OF UNIFORM STRENGTH

Consider a flat rotating disc of uniform strength σ at all radii.

Taking $\sigma_1 = \sigma_2 = \sigma$

Let t be the thickness at a radius r and $t + \delta t$ at a radius $r + \delta r$ (Fig. 14.10),

Mass of element $= \rho \cdot r\delta\theta \cdot \delta r \cdot t$

Centrifugal force $= (\rho \cdot r\delta\theta \cdot \delta r \cdot t)r\omega^2$ (Outward)

Radial force on inner face $= \sigma_r\,(r \cdot \delta\theta)t$ (Inward)

Radial force on outer face $= \sigma(r + \delta r) \cdot \delta\theta \cdot (t + \delta t)$ (Outward)

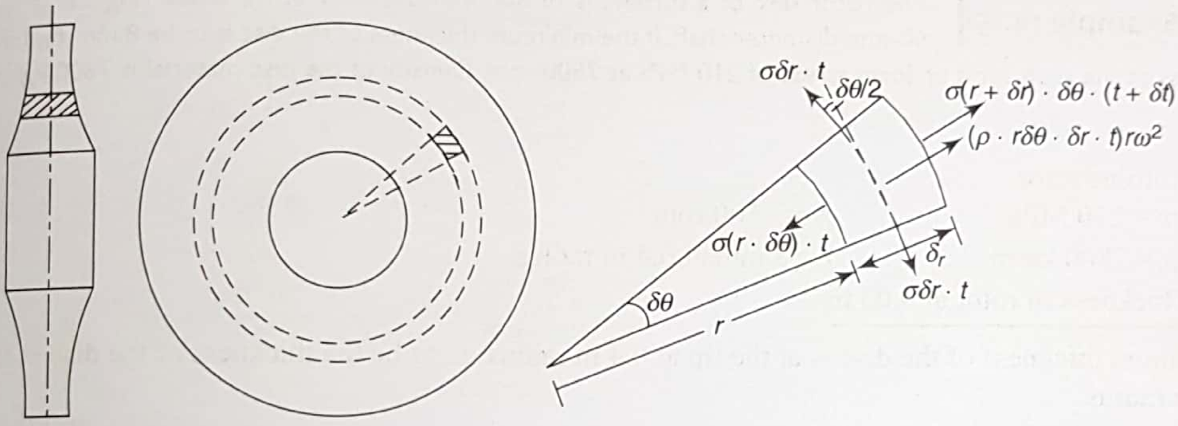


Fig. 14.11

Radial components of tangential force $= 2\sigma \cdot \delta r \cdot t \sin \frac{1}{2}\delta\theta \approx \sigma \cdot \delta r \cdot t \cdot \delta\theta$ (Inward)

For equilibrium,

$$\sigma \cdot \delta r \cdot t \cdot \delta\theta + \sigma \cdot r \cdot \delta\theta \cdot t - \sigma(r + \delta r) \cdot \delta\theta(t + \delta t) = \rho \cdot r^2 \cdot \delta\theta \cdot \delta r \cdot t \cdot \omega^2$$

$$\sigma \cdot \delta r \cdot t + \sigma \cdot r \cdot t - \sigma(r + \delta r)(t + \delta t) = \rho r^2 \cdot \delta r \cdot t \cdot \omega^2$$

Simplifying and taking limits,

$$\sigma \cdot t \cdot dr + \sigma \cdot r \cdot t - \sigma \cdot r \cdot t - \sigma \cdot r \cdot dt - \sigma \cdot t \cdot dr = \rho r^2 \omega^2 \cdot t \cdot dr$$

$$\sigma \cdot \frac{dt}{t} = -\rho \cdot \omega^2 \cdot r \cdot dr$$

Integrating, $\log t = -\dfrac{\rho \cdot \omega^2}{\sigma} \cdot \dfrac{r^2}{2} + \text{constant}$

$$t = A \cdot e^{-\rho \cdot \omega^2 \cdot r^2/2\sigma} \qquad (14.30)$$

Let $t = t_o$ at $r = 0$, $\therefore A = t_o$

Thus

$$t = t_o \cdot e^{-\rho \cdot \omega^2 \cdot r^2/2\sigma} \qquad (14.31)$$

Example 14.14 A steam turbine rotor designed for uniform strength of 70 MPa rotates at 3600 rpm. The thickness of the rotor at the centre is 20 mm. Determine the thickness of the rotor at a radius of 400 mm. Density of the material of the rotor is 7700 kg/m³.

Solution

Given A turbine rotor

$t_o = 20$ mm $\qquad \sigma = 70$ MPa

$N = 3600$ rpm $\qquad r = 400$ mm

$\rho = 7700$ kg/m³

To find Thickness of rotor

Using Eq. 14.31,

$$t = t_o \cdot e^{-\rho \cdot \omega^2 \cdot r^2 / 2\sigma}$$

where $$\frac{\rho\omega^2 r^2}{2\sigma} = 7700 \times \left(\frac{2\pi \times 3600}{60}\right)^2 \times \frac{0.4^2}{2 \times 70 \times 10^6} = 1.2507$$

Thickness of rotor

$$t = t_o \cdot e^{-1.2507} = 20 \times e^{-1.2507} = 5.726 \text{ mm}$$

Example 14.15 The rotor disc of a turbine is of 800-mm diameter at the blade ring and is fixed to a 60-mm diameter shaft. If the minimum thickness of the disc is to be 8 mm, find the thickness at the shaft for a uniform stress of 210 MPa at 7500 rpm. Density of the disc material is 7800 kg/m^3.

Solution

Given A turbine rotor

$\sigma = 210$ MPa $\qquad N = 7500$ rpm

$\rho = 7800$ kg/m^3 $\qquad t = 8$ mm at 0.4 m radius

To find Thickness of rotor at 0.03 m

The minimum thickness of the disc is at the tip at 0.4 m radius. Let t be the thickness of the disc at the shaft at 0.03 m radius.

Now as $$t = A \cdot e^{-\rho \cdot \omega^2 \cdot r^2 / 2\sigma}$$

At* 0.4 *m radius

$$8 = A \cdot e^{-\rho \cdot \omega^2 \cdot 0.4^2 / 2\sigma} \qquad \text{(i)}$$

At* 0.03 *m radius

$$t = A \cdot e^{-\rho \cdot \omega^2 \cdot 0.03^2 / 2\sigma} \qquad \text{(ii)}$$

Dividing (ii) and (i),

$$t = 8 \times \frac{A \cdot e^{-\rho \cdot \omega^2 \cdot 0.03^2 / 2\sigma}}{A \cdot e^{-\rho \cdot \omega^2 \cdot 0.4^2 / 2\sigma}} = 8 \times e^{\rho \cdot \omega^2 (0.16 - 0.0009)/2\sigma} = 8 \times e^{0.1591 \rho \cdot \omega^2 / 2\sigma}$$

where $$\frac{0.1591\rho\omega^2}{2\sigma} = 7800 \times \left(\frac{2\pi \times 7500}{60}\right)^2 \times \frac{0.1591}{2 \times 210 \times 10^6} = 1.8226$$

$\therefore$ $$t = 8e^{1.8226} = 49.5 \text{ mm}$$

6.6 SHEAR CENTRE

LO 6

Define shear centre and to determine the same for different sections

So far, the analysis for the shear stresses due to transverse loading was limited to members having a vertical plane of symmetry. The load is applied in that plane and the member is observed to bend in the plane of loading. Consider the case of a channel with its web in the horizontal position (Fig. 6.31). When it is loaded through its vertical line of symmetry, the vertical shear force is taken by two flanges only and web is considered to take only a marginal part of vertical shear force which is ignored. Thus, vertical force in each flange is $F/2$. The transverse shear force in flanges will be negligible and in the web it is symmetrically distributed about the vertical line of symmetry in opposite directions.

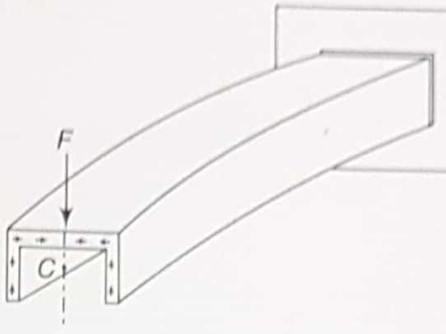
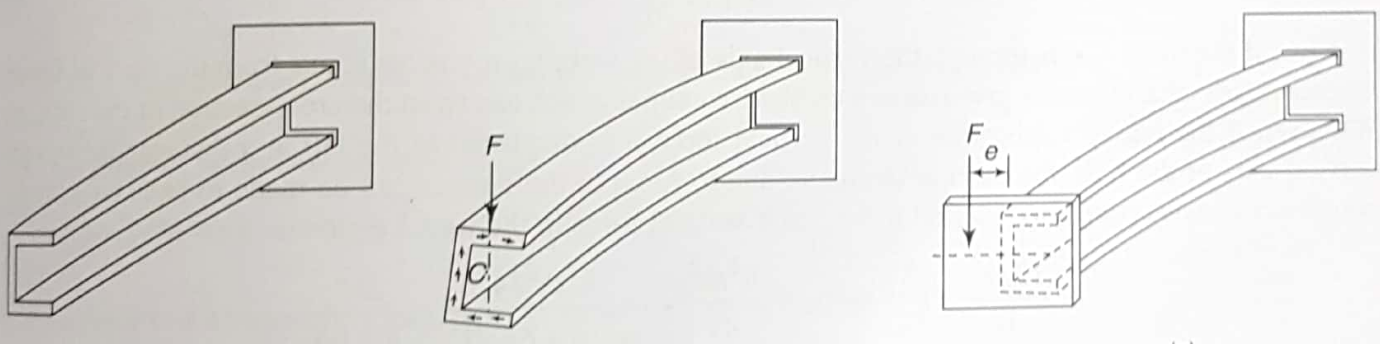


Fig. 6.31

However, there can be transverse loading on thin-walled members which do not have vertical plane of symmetry. In Fig. 6.32a the channel is turned through 90°. Now, though the line of action of the loading may still be passing through the centroid of the end section, the member is observed to bend and twist under the loading because now the vertical shear force is taken by the web only and the vertical shear force taken by flanges is negligible (Fig. 6.32b). The transverse shear force in the web is negligible but in the flanges they form a couple formed by horizontal forces in the upper and lower legs of the channel. In such cases, the channel section will twist unless the line of action of load is displaced as shown in Fig. 6.32c.



(a) (b) (c)

Fig. 6.32

Shear centre is the point in or outside a section through which the shear force applied produces no torsion or twist of the member.

- For a beam with two axes of symmetry, the shear centre coincides with the centroid.
- For sections having one axis of symmetry, shear centre does not coincide with the centroid, though it lies on the axis of symmetry.

Consider the channel section shown in Fig. 6.33a. Assuming uniform thickness t of the web and flanges,

$$\text{Moment of inertia, } I_x = \frac{t \times h^3}{12} + 2\left[\frac{b \times t^3}{12} + bt\left(\frac{h}{2}\right)^2\right] \approx \frac{th^2}{12}(h + 6b)$$

(Neglecting moment of inertia of flanges about their axes, i.e., neglecting the first term in the bracket)

Consider an elementary length dz of the flange at a distance z from the tip (Fig. 6.33b),

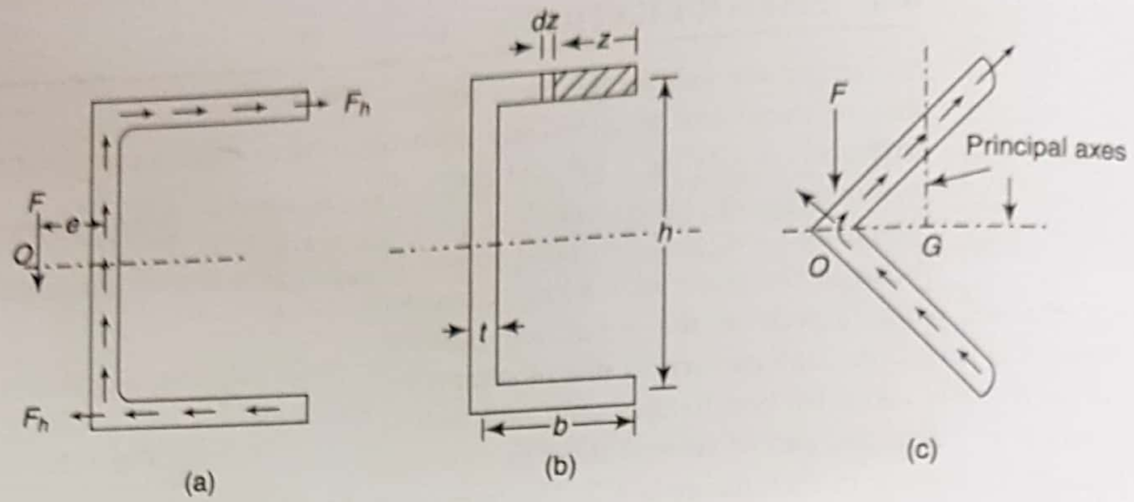


Fig. 6.33

Shear stress in the elementary length, $\tau = F \cdot \dfrac{(tz)\bar{y}}{tI} = F \cdot \dfrac{z}{I} \cdot \dfrac{h}{2} = \dfrac{Fzh}{2I}$

Shear force in the elementary length $= \tau \times$ area $= \dfrac{Fzh}{2I} \cdot (t \cdot dz) = \dfrac{Fht}{2I} \cdot (z \cdot dz)$

Total force in each flange, $F_f = \dfrac{Fht}{2I} \int_0^b z \cdot dz = \dfrac{Fht}{2I} \left[\dfrac{z^2}{2} \right]_0^b = \dfrac{Fhtb^2}{4I}$ (6.20)

Now, if the force F acts through the vertical axis of the web, no moments result from the vertical forces whereas clockwise moments give rise to a clockwise couple which can twist the cross-section of the channel. However, if the line of application of the vertical force F is displaced to the left at a distance e from the vertical axis of the web, the clockwise couple due to force in the flanges can be made to balance with the counter-clockwise couple to external force F and vertical force in the web, i.e.,

$$\frac{Fhtb^2}{4I} \times h \quad \text{or} \quad e = \frac{h^2tb^2}{4I} = \frac{h^2tb^2/4}{th^2(h+6b)/12} = \frac{3b^2}{h+6b}$$

In case of an equal angle section, the principal axes are as shown in Fig. 6.33c. As the shear forces due to shear stresses in the two legs as well as the external forces intersect at a common point O, no resultant moments are obtained of any force and thus the point O is the shear centre.

Example 6.13 Determine the position of the shear centre for an 80 mm by 40 mm outside by 5-mm thick channel section.

Solution

Given A channel section as shown in Fig. 6.34

To find To locate shear centre

Refer Fig. 6.34,

Moment of inertia about x-axis

$$I_x = \frac{5 \times 80^3}{12} + 2\left[\frac{35 \times 5^3}{12} + 35 \times 5 \times 37.5^2 \right] = 706\,250 \text{ mm}^4$$

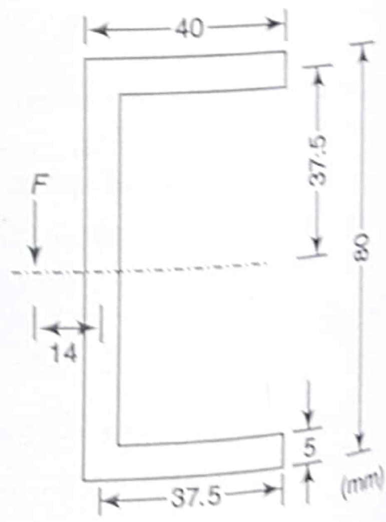


Fig. 6.34

Total shear force in the flanges

$$F_f = \frac{Fhtb^2}{4I} = \frac{F \times 75 \times 5 \times 37.5^2}{4 \times 706\ 250} = 0.186\ 68\ F \qquad \text{...(Eq. 6.20)}$$

Calculation for location of shear centre

If e is the distance of the shear centre from the centre line of the web, then for equilibrium,

$$F \cdot e = F_f \times 2\bar{y} \quad \text{or} \quad F \cdot e = 0.18668F \times 2 \times 37.5 \quad \text{or} \quad e = 14 \text{ mm}$$

or by the relation, $e = \dfrac{3b^2}{h+6b} = \dfrac{3(37.5)^2}{75+6(37.5)} = 14.06$ mm

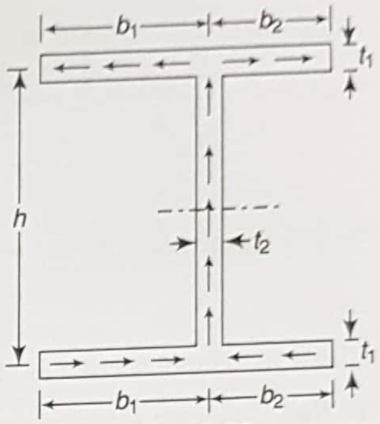


Fig. 6.35

Example 6.14 Locate the shear centre for the section shown in Fig. 6.35.

Solution

Given An I-section as shown in Fig. 6.35

To find To locate shear centre

Refer Fig. 6.36,

Moment of inertia about x-axis

$$I_x = 2\left[\frac{t_2h^3}{12} + (b_1+b_2)\frac{t_1^3}{12} + (b_1+b_2)t_1\left(\frac{h}{2}\right)^2\right]$$

Consider an elementary length dz of the left part of flange at a distance z from the tip.

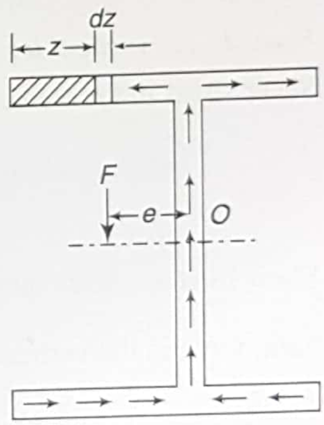


Fig. 6.36

Shear force in elementary length

Shear stress in the elementary length, $\tau = F \times \dfrac{(tz)\bar{y}}{tI} = F \times \dfrac{(t_1z)}{t_1I} \times \dfrac{h}{2} = \dfrac{Fzh}{2I}$

Shear force in the elementary length $= \tau \times$ area $= \dfrac{Fzh}{2I} \times (t_1 \times dz) = \dfrac{Fht_1}{2I} \cdot (z \cdot dz)$

Total force in left part of flange, $F_f = \dfrac{Fht_1}{2I}\displaystyle\int_0^{b_1} z \cdot dz = \dfrac{Fht_1}{2I}\left[\dfrac{z^2}{2}\right]_0^{b_1} = \dfrac{Fht_1b_1^2}{4I}$

Similarly, Total force in right part of flange, $F_f = \dfrac{Fht_1b_2^2}{4I}$

Calculation for location of shear centre

Taking moments of the shear forces about the web centre O,

$$F \cdot e = \frac{Fht_1b_1^2}{4I1} \times h - \frac{Fht_1b_2^2}{4I} \times h = \frac{Fht_1h}{4I}(b_1^2 - b_2^2)$$

$$e = \frac{h^2t_1}{4I}(b_1^2 - b_2^2)$$

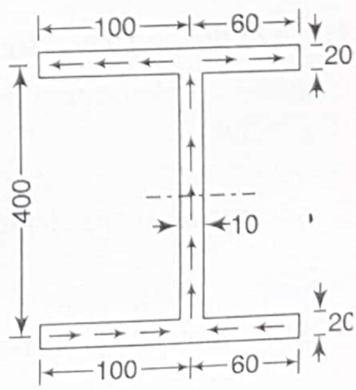


Fig. 6.37

Example 6.15 Locate the shear centre for the section shown in Fig. 6.37.

Solution

Given An I-section as shown in Fig. 6.37

To find To locate shear centre

Moment of inertia about x-axis

$$I_x = 2\left[\frac{10\times 380^3}{12} + (100+60)\frac{20^3}{12} + (100+60)\times 20\times 200^2\right]$$

$$= 2(45.73 + 0.107 + 128)\times 10^6 = 173.84\times 10^6\ \text{mm}^4$$

Calculation for location of shear centre

Using the result obtained in the previous example,

$$e = \frac{h^2 t_1}{4I}(b_1^2 - b_2^2) = \frac{400^2\times 20}{4\times 173.84\times 10^6}(100^2 - 60^2) = 29.45\ \text{mm}$$

Example 6.16 Locate the shear centre for the section shown in Fig. 6.38.

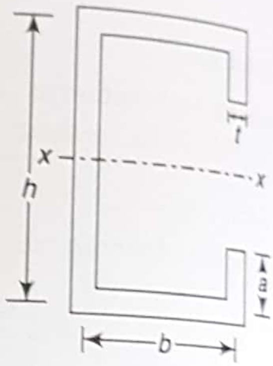


Fig. 6.38

Solution

Given A channel section as shown in Fig. 6.38

To find To locate shear centre

Consider an elementary length dy of the vertical leg of the flange at a distance y from the tip (Fig. 6.39a).

Force in the vertical leg of flanges

Shear stress in the elementary length,

$$\tau = F\cdot\frac{(ty)\bar{y}}{tI} = F\cdot\frac{y}{I}\cdot\left[\frac{h}{2} - a + \frac{y}{2}\right] = \frac{Fy}{2I}\cdot(h - 2a + y)$$

Shear force in the elementary length $= \tau \times$ area $= \frac{Fy}{2I}\cdot(h-2a+y)\cdot t\,dy$

Total force in the vertical leg of each flange,

$$F_1 = \frac{Ft}{2I}\int_0^a (hy - 2ay + y^2)\,dy = \frac{Ft}{2I}\left[h\frac{y^2}{2} - 2a\frac{y^2}{2} + \frac{y^3}{3}\right]_0^a = \frac{Fta^2}{12I}(3h - 4a)$$

Force in the horizontal leg of flanges

Consider an elementary length dz of the horizontal leg of the flange at a distance z from the right end (Fig. 6.39b).

Shear stress in the elementary length, $\tau = \frac{F}{tI}\left[at\left(\frac{h}{2} - \frac{a}{2}\right) + tz\cdot\frac{h}{2}\right]$

Shear force in the elementary length $= \tau \times$ area $= \frac{F}{tI}\left[at\left(\frac{h}{2} - \frac{a}{2}\right) + tz\cdot\frac{h}{2}\right](t\cdot dz)$

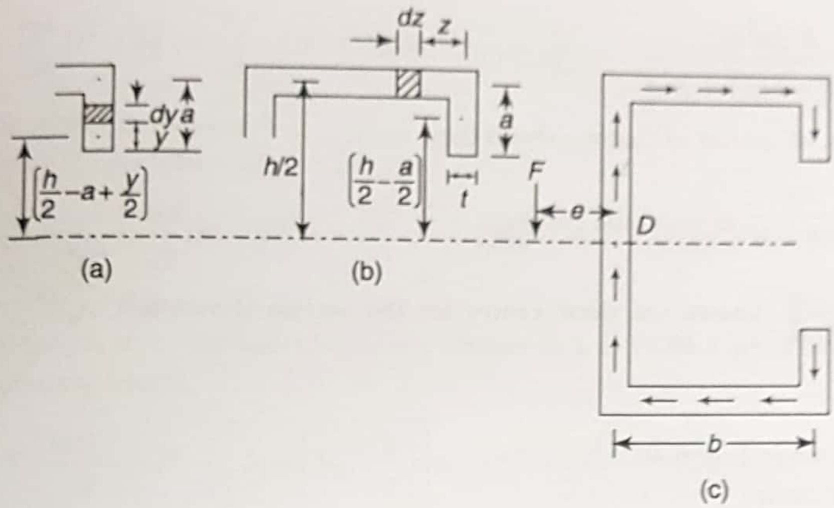


Fig. 6.39

Total force in horizontal leg of each flange,

$$F_2 = \frac{F}{tI}\int\left[at\left(\frac{h}{2}-\frac{a}{2}\right)+tz\cdot\frac{h}{2}\right]\cdot tdz = \frac{Ft}{2I}\int_0^b a((h-a)+hz)dz$$

$$= \frac{Ft}{2I}\left[a(h-a)\cdot z + h\frac{z^2}{2}\right]_0^b = \frac{Ftb}{4I}[2a(h-a)+hb]$$

Calculation for location of shear centre

Taking moments about the point D,

$$F\cdot e = 2F_1\cdot b + 2F_2\cdot\frac{h}{2} = 2F_1\cdot b + F_2\cdot h = 2\frac{Fta^2}{12I}(3h-4a)\cdot b + \frac{Ftb}{4I}[2a(h-a)+hb]h$$

$$e = \frac{bt}{12I}(6ha^2 - 8a^3 + 6ah^2 - 6ha^2 + 3h^2b)$$

$$= \frac{bt}{12I}(6ah^2 - 8a^3 + 3h^2b)$$

Moment of inertia

$$I_x = 2\left[\frac{ta^3}{12}+ta\left(\frac{h}{2}-\frac{a}{2}\right)^2\right]+2\left[\frac{bt^3}{12}+bt\left(\frac{h}{2}\right)^2\right]+\frac{th^3}{12}$$

$$= t\left[\frac{a^3}{6}+\frac{a}{2}(h^2-2ah+a^2)+\frac{bt^2}{6}+\frac{bh^2}{2}+\frac{h^2}{12}\right]$$

$$= \frac{t}{12}[8a^3 + 6ah^2 - 12a^2h + 2bt^2 + 6bh^2 + h^3]$$

Shear centre

$$e = \frac{b(6ah^2 - 8a^3 + 3h^2b)}{(8a^3 + 6ah^2 - 12a^2h + 2bt^2 + 6bh^2 + h^3)} \qquad \text{(Fig. 6.39c)}$$