Control Systems

Unit I: Introduction

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Introduction

Control engineering deals with the design (and implementation) of control systems using linear, time-invariant mathematical models representing actual physical nonlinear, time-varying systems with parameter uncertainties in the presence of external disturbances.

Need for Control System

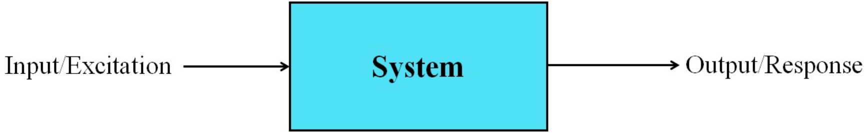
1. Power amplification
2. Remote control
3. Convenience of input form
4. Compensation from disturbances

Introduction

Control system engineering is based on the foundations of feedback theory and linear system analysis, and it integrates the concepts of network theory and communication theory. Indeed, control engineering is not limited to any engineering discipline but is equally applicable to aerospace, agricultural, biomedical, chemical, civil, computer, industrial, electrical, environmental, mechanical, and nuclear engineering. Many aspects of control engineering can also be found in studies in systems engineering.

The face of control engineering is rapidly changing. The coming age of the Internet of Things (IoT) presents many intriguing challenges in control system applications in the environment (think about more efficient energy use in homes and businesses), manufacturing (think 3D printing), consumer products, energy, medical devices and healthcare, transportation (think about automated cars!), among many others

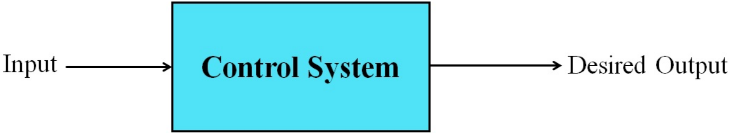
What is a System?
A collection of physical, biological or abstract components which together perform an intended objective
A system gives an output (also called response) for an input (also called excitation)



System can be a collection of multiple sub-systems

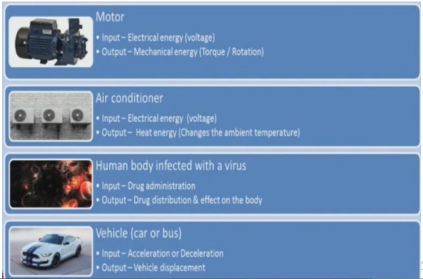
Introduction about concepts of plant, system and control system

What is a Control?
The term control means to regulate, to direct or to command
Combining above two definitions
Control+System=Control System
A control system is defined as a combination of devices and components connected or related so as to command, direct or regulate itself or another system.

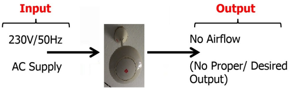


Introduction about concepts of plant, system and control system

The part of the system which is to be controlled is given different names, for example, plant, controlled system, process, etc. The control system designer develops a controller that will control the plant or the controlled system.
Control systems are used in many applications, for example, the control of temperature, liquid level, position, velocity, flow, pressure, acceleration, etc.



A Fan without blades cannot be a SYSTEM, because it cannot provide a desired/proper output i.e. airflow

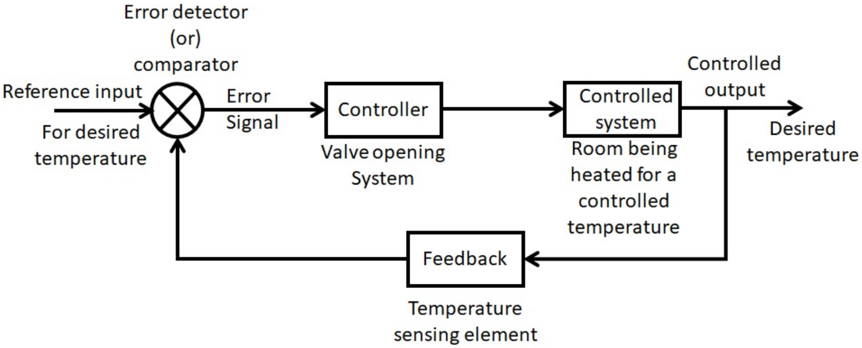


A Fan with blades but without regulator can be a SYSTEM, because it can provide a proper output i.e. airflow

A Fan with blades and with regulator can be a CONTROL SYSTEM, because it can provide a Desired output. i.e. Controlled airflow

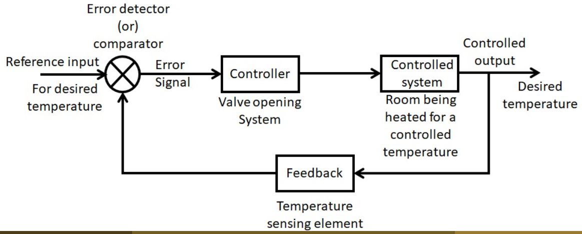
Introduction about concepts of plant, system and control system

The basic components of a control system forming a control diagram are shown in Figure. For easy understanding, consider room heating system using steam flowing through pipes fitted in the room.



Introduction about concepts of plant, system and control system

The flow of steam through the pipes is regulated automatically by a control valve. The amount of opening of the valve is regulated by a servomotor. The servomotor, controlling the opening or closing of the valve, works as a controller for amount of heating of the room. A temperature sensor will measure the room temperature and will provide a feedback signal for comparison in the comparator. A reference input is provided to the comparator for the desired temperature. In case the output temperature is different from the desired temperature, an error signal will be generated to control the hot steam flow through the valve.



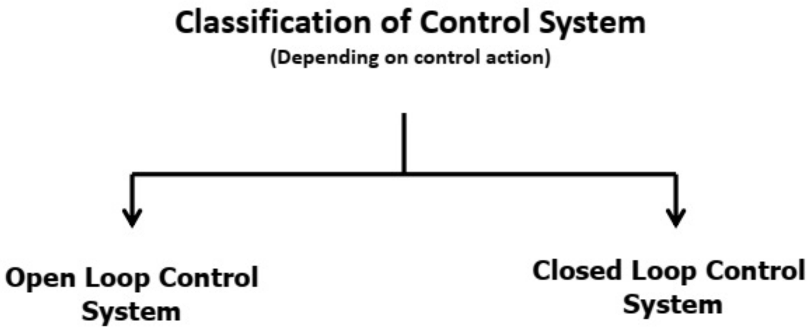
Introduction about concepts of plant, system and control system

The components shown in the diagram above are defined as follows:
1. Reference Input: This provides input signal for the desired output.
2. Error Detector: It is an element in which one system variable (feedback signal) is subtracted from another variable (reference signal) to obtain a third variable (error signal). It is also called comparator.
3. Feedback Element: Feedback signal is a function of the controlled output which is compared with the reference signal to obtain the error or the actuating signal. Feedback element measures the controlled output, converts or transforms to a suitable value for comparison with the reference input.
4. Error Signal: It is an algebraic sum of the reference input and the feedback.

Introduction about concepts of plant, system and control system

5. Controller: The controller is an element that is required to generate the appropriate control signal. The controller operates until the error between the controlled output and desired output is reduced to zero.
6. Controlled System: It is a body, a plant, a process or a machine of which a particular condition is to be controlled, for example, a room heating system, a spacecraft, reactor, boiler, CNC machine, etc.
7. Controlled Output: Controlled output is produced by the actuating signal available as input to the controller. Controlled output is made equal to the desired output with the help of the feedback system.

Classification of control systems



Examples for Open loop control systems

Definition:

"A system in which the control action is totally independent of the output of the system is called as open loop system"

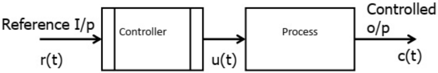


Fig. Block Diagram of Open loop Control System

An actuator is a device employed by the control system to alter or adjust the environment.

- **Electric hand drier** – Hot air (output) comes out as long as you keep your hand under the machine, irrespective of how much your hand is dried.

Examples for Open loop control systems

➢ **Automatic washing machine**

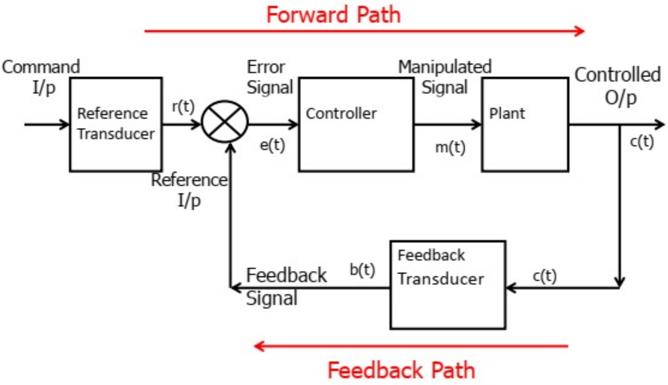
– This machine runs according to the pre-set time irrespective of washing is completed or not.

➢ **Bread toaster** - This machine runs as per adjusted time irrespective of toasting is completed or not.

Examples for Open loop control systems

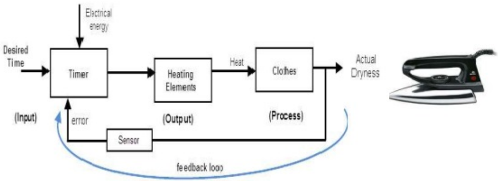
- **Automatic tea/coffee Vending Machine** – These machines also function for pre adjusted time only.

Definition:
A system in which the control action is somehow dependent on the output is called as closed loop system



A sensor is a device that provides a measurement of a desired external signal. For example, resistance temperature detectors (RTDs) are

- **Automatic Electric Iron**- Heating elements are controlled by output temperature of the iron.



- **Servo voltage stabilizer** – Voltage controller operates depending upon output voltage of the system.

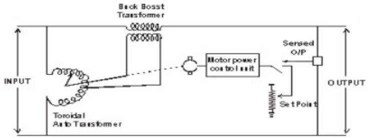


Fig. 5.6 Servo Voltage Stabilizer

Advantages of OLCS

- Simple in construction and design.
- Economical.
- Easy to maintain.
- Generally stable.
- Convenient to use as output is difficult to measure.

Disadvantages of OLCS

- They are inaccurate
- They are unreliable
- Any change in output cannot be corrected automatically.

Advantages of CLCS

- Closed loop control systems are more accurate even in the presence of non-linearity.
- Highly accurate as any error arising is corrected due to presence of feedback signal.
- Bandwidth range is large.
- Facilitates automation.
- The sensitivity of system may be made small to make system more stable.
- This system is less affected by noise.

Disadvantages of CLCS

- They are costlier.
- They are complicated to design.
- Required more maintenance.
- Feedback leads to oscillatory response.
- Overall gain is reduced due to presence of feedback.
- Stability is the major problem and more care is needed to design a stable closed loop system.

Differences between OLCS and CLCS

<u>Open Loop Control System</u>	<u>Closed Loop Control System</u>
1. The open loop systems are simple & economical.	1. The closed loop systems are complex and costlier
2. They consume less power.	2. They consume more power.
3. The OL systems are easier to construct because of less number of components required.	3. The CL systems are not easy to construct because of more number of components required.
4. The open loop systems are inaccurate & unreliable	4. The closed loop systems are accurate & more reliable.

Differences between OLCS and CLCS

Open Loop Control System	Closed Loop Control System
5. Stability is not a major problem in OL control systems. Generally OL systems are stable.	5. Stability is a major problem in closed loop systems & more care is needed to design a stable closed loop system.
6. Small bandwidth.	6. Large bandwidth.
7. Feedback element is absent.	7. Feedback element is present.
8. Output measurement is not necessary.	8. Output measurement is necessary.

Differences between OLCS and CLCS

Open Loop Control System

9. The changes in the output due to external disturbances are not corrected automatically. So they are more sensitive to noise and other disturbances.

10. Examples:

    Coffee Maker,

    Automatic Toaster,

    Hand Drier.

Closed Loop Control System

9.The changes in the output due to external disturbances are corrected automatically. So they are less sensitive to noise and other disturbances.

10. Examples:

    Guided Missile,

    Temp control of oven,

    Servo voltage stabilizer.

Classification of Control Systems

Based on the nature of systems, control systems is further classified into linear and non-linear, static or dynamic, time variant or time-invariant systems and causal or non-causal systems.

When an input X1 produces an output Y1 and an input X2 produces an output Y2, then any combination aX1+bX2 should produce an output aY1+bY2. In such case system is linear.
Therefore, linear systems are those where the principles of superposition and proportionality are obeyed.

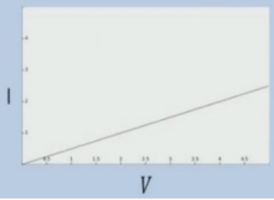
Non-linear systems do not obey law of superposition.
The stability of non-linear systems depends on root location as well as initial conditions and type of input.
Non-linear systems exhibit self sustained oscillations of fixed frequency.

Example for Linear or Non-Linear systems

Linear Vs Non-Linear Systems

Linear systems

- Output of the system varies linearly with input
- Satisfy homogeneity and superposition
- E.g. Resistor : $I = \frac{V}{R}$



Non-linear systems

- Output of the system does not vary linearly with input
- Do not satisfy homogeneity and superposition
- E.g. Diode: $I = I_0(e^{\frac{V}{\tau}} - 1)$

Differences between Linear and Non-Linear control systems

Linear System	Non-linear System
1. Obey superposition.	1. Do not obey superposition
2. Can be analyzed by standard test signals	2. Cannot be analyzed by standard test signals
3. Stability depends only on root location	3. Stability depends on root locations, initial conditions & type of input
4. Do not exhibit limit cycles	4. Exhibits limit cycles
5. Do not exhibit hysteresis/ jump resonance	5. Exhibits hysteresis/ jump resonance
6. Can be analyzed by Laplace transform, z- transform	6. Cannot be analyzed by Laplace transform, z- transform

Example for Static or Dynamic Control Systems

Static Vs Dynamic Systems

Static systems

- At any time, output of the system depends only on present input
- Memory less systems
- $y(t) = f(u(t))$
- E.g. Resistor:

$$I(t) = \frac{V(t)}{R}$$

Dynamic systems

- Output of the system depends on present as well as past inputs
- Presence of memory can be observed
- $y(t) = f(u(t), u(t-1), u(t-2), \ldots)$
- E.g. Inductor:

$$I(t) = \frac{1}{L}\int_0^t V(t)dt$$

Time varying and in-varying Control Systems

Systems whose parameters vary with time are called time varying control systems.
When parameters do not vary with time are called Time Invariant control systems.
The mass of missile/rocket reduces as fuel is burnt and hence the parameter mass is time varying and the control system is time varying type.

Example for Causal or Non-causal control systems

Causal Vs Non-causal Systems

Causal systems

- Output is only dependent on inputs already received (present or past)
- Non-anticipatory system
- $y(t) = f(x(t), x(t-1), \ldots)$
- E.g.
  - Thermostat based AC
  - Motor or generator

Non-causal systems

- Output depends on future inputs as well
- System anticipates future inputs based on past
- $y(t) = f(x(t), x(t+1), \ldots)$
- E.g.
  - Weather forecasting system
  - Missile guidance system

Servomechanism, Regulator, Process control and Disturbance signal

Servomechanism is an automatic control system in which the controlled variable value is forced to follow the variations of reference value, instead of regulating a variable value to "set point". For example, control of an industrial robot arm, a position control system, etc. It is also called tracking control system.
Regulator is a feedback control system in which controlled variable is maintained at a constant value inspite of external load on the plant. Examples are regulation of steam supply in steam engine by fl y-ball governor, thermostat control of home heating systems, regulation of the voltage of an alternator.

Servomechanism, Regulator, Process control and Disturbance signal

Process control refers to control of such parameters as level, flow, pressure, temperature and acidity of process variables.
Disturbance represents the undesired signals that tend to affect the controlled system. Disturbance may be due to changes made in set point, amplifi er noise, variation in load, wind power disturbing outdoor installation, etc. Other disturbances that affect the performance of the control system may be changes in parameters due to wear, ageing, environmental effects,high frequency noise introduced by the measurement sensors, etc. Sometimes disturbance signals may be too fast for the control system to take care of. Low pass fi lters may be used to take care of high frequency disturbance signals thereby maintaining satisfactory performance of the control system.

Feedback in control system and effect oF Feedback

Feedback control is now a basic feature of modern industry. In present-day technological society, in order to utilize natural resources optimally, some form of control is needed.

Control engineering is primarily concerned with controlling industrial processes and natural resources, and forces of nature purposefully and for the benefit of mankind.

Early machine and equipment used for control were primarily, manually operated type, requiring frequent adjustment so as to maintain and/or achieve the desired performance.

Advanced technology made revolutions in the procedure used for system analysis and design.

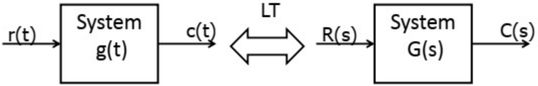
Use of feedback in control systems brings in significant changes in terms of improvement in overall gain, improvement in system stability, reduction of sensitivity of the system to variations in system parameters, and neutralizing or reducing the effect of disturbance signals.

Time varying control systems MODELLING A CONTROL SYSTEM—TRANSFER FUNCTION APPROACH

A control system consists of a number of sub-systems. All the sub-systems work in unison to achieve a desired output for a given input. Mathematical modelling of sub-systems and also of the whole system is required for carrying out performance studies. So, modelling of a control system using transfer function approach has been dealt with. State space model as well.

TRANSFER FUNCTION

The relationship between input and output of a linear, time-invariant system is given by the transfer function.
Definition: The ratio of Laplace transform of the output to the Laplace transform of the input under the assumption of zero initial conditions is defined as Transfer Function.



For the system shown,

c(t)= output
r(t)= input
g(t)= System function

L{c(t)}= C(s)
L{r(t)}= R(s)
L{g(t)}= G(s)

Therefore transfer function G(s) for above system is given by,

$$G(s) = \frac{\text{Laplace of output}}{\text{Laplace of input}} = \frac{C(s)}{R(s)}$$

Procedure For Determining the transfer Function of a control system

Transfer function of any system can be determined through the following steps:
Step 1: Formulate the mathematical equation for the system.
Step 2: Take the Laplace transform of the system equation assuming all the initial conditions of the system as zero.
Step 3: Take the ratio of Laplace transform of the output to the Laplace transform of the input.
Let us take an example. Let a system be described by the following differential equation:
Step 1: We write the equation representing the system

$$5\ddot{y} + 3\ddot{y} + 2\dot{y} + y = 4\ddot{x} + 2\dot{x} + x$$

where y is the system output and x is the system input.

Procedure For Determining the transfer Function of a control system

Step 2: Take the Laplace transform of above equation. By assuming all the initial conditions as zero, we get

$$\left(5s^3 + 3s^2 + 2s + 1\right) Y(s) = \left(4s^2 + 2s + 1\right) X(s)$$

Step 3: Take the ratio of Laplace transform of the output to the Laplace transform of the input,

$$\text{Transfer function, } G(s) = \frac{Y(s)}{X(s)} = \frac{4s^2 + 2s + 1}{5s^3 + 3s^2 + 2s + 1}$$

After factorization of the numerator and the denominator of the transfer function, the poles and zeros can be determined.

Formulation of equations of Physical systems and their transfer Functions

A physical system consists of a number of sub-systems connected together to serve a specific purpose. For example a motor car as a mechanical system, it has a number of sub-systems like ignition sub-system, pneumatic sub-system, power transmission sub-system, and so on. Similarly, electrical systems. mechanical systems, rotational systems, translational systems, and so on.
For better understanding about the performance of a control system it is convenient to develop mathematical models of such systems and study and modify them for giving better performance. But, almost all physical systems are non-linear to some extent due to that it may be difficult to write exact mathematical equations for all systems. Therefore, it is necessary to use the best possible linear approximation to analyse such systems.

Formulation of equations of Electrical Systems

A resistor, an inductor and a capacitor are the three basic elements of an electric circuit. The circuit is analysed by the application of Kirchhoff's voltage and current laws.

The relationship that exists between voltage and the current flowing through the circuit elements may be expressed as: For a resistive circuit,

$$v = iR \qquad ......(1.1)$$

Taking Laplace transform,

$$V(s) = RI(s) \qquad ......(1.2)$$

For a capacitive circuit, $v = \frac{1}{C}\int_o^t idt$ ......(1.3)

(Considering initial conditions to be zero) Taking Laplace transform of equation 1.3,

$$V(s) = \frac{1}{Cs}I(s) \qquad ......(1.4)$$

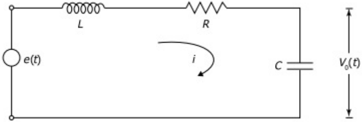
For an inductive circuit,

$$v = L\frac{di}{dt} \qquad ......(1.5)$$

Taking Laplace transform of equation 1.5,

$$V(s) = sLI(s) \qquad ......(1.6)$$

When these basic elements form an electrical circuit, mathematical formulation is made by using Kirchhoff's laws. The RLC circuit of Fig. is analysed by Kirchhoff's voltage law applied to the closed loop.



The system equation is

$$e(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt \qquad ......(1.7)$$

Now taking Laplace transform on both sides, we get

$$E(s) = \mathrm{Ls}\, I(s) + RI(s) + \frac{1}{Cs}I(s) \qquad ......(1.8)$$

(assuming all initial conditions to be zero)

$$E(s) = \left[Ls + R + \frac{1}{Cs}\right] I(s) \qquad ......(1.9)$$

$$E(s) = \left[\frac{LCs^2 + RCs + 1}{Cs}\right] I(s) \qquad ......(1.10)$$

Let the output voltage $v_o(t)$ be taken across the capacitor, C. Then,

$$v_o(t) = \frac{1}{C}\int idt \qquad ......(1.11)$$

Taking Laplace transform on both sides of equation (1.11), we get

$$V_0(s) = \frac{1}{Cs} I(s) \qquad ......(1.12)$$

(assuming all initial conditions to be zero) Therefore, the transfer function is given by

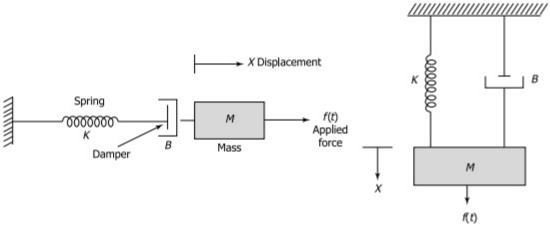
$$G(s) = \frac{V_o(s)}{E(s)} \qquad ......(1.13)$$

$$= \frac{\frac{1}{Cs}}{(LCs^2 + RCs + 1)} \cdot Cs \frac{I(s)}{I(s)}$$

$$\text{Transfer function, } G(s) = \frac{V_o(s)}{E(s)} = \frac{1}{(LCs^2 + RCs + 1)} \qquad ......(1.14)$$

The block diagram representation shown in Fig.

In analysis of mechanical systems use three idealized elements, namely, inertial elements such as mass or moment of inertia, spring and a damper. The inertial elements, i.e. mass and spring are capable of storing energy while the damper is capable of dissipating energy. A damper is often referred to as mechanical resistance. A mechanical system may have either purely translational motion or purely rotational motion.



The equations of motion are generally formulated using Newton's laws of motion. In pure translational systems, motion is considered to take place in straight lines. Therefore, the variables-force, displacement, velocity and acceleration-are aligned in a straight line. If a mass m moves in a straight line with x as its position at any time with reference to a fixed reference axis, then by applying Newton's laws of motion, the force F acting upon it may be expressed as

$$\text{Force} = \text{Mass} \times \text{Acceleration; i.e. } F = m\frac{d^2x}{dt^2} = m\ddot{x} \qquad ......(1.15)$$

Again, if a mass m is moving with a velocity v, it will have a stored translational energy of $1/2mv^2$. This provides us with an idea that mass is an energy storing device.

A spring, when acted upon by a force, gets stressed. For a linear spring, the deformation produced is directly proportional to the magnitude of the applied force. The equation describing the relationship of F and x for a linear spring is

$$F = Kx \qquad ......(1.16)$$

where F is the force exerted on the spring, x is the deformation of the spring, and K is the stiffness coefficient of the spring.
A spring is usually represented by a coil. A spring is compressed or elongated by an amount x due to a force F.

A damper is generally represented by a piston in a cylinder. A damper produces resistance velocity which is often termed as friction. In a fluid medium this friction is due to the viscosity. When the force due to friction is proportional to relative velocity, the friction is known as linear friction. The relationship of forces in a linear damper is given by

$$f(t) = B\frac{dx}{dt} = B\dot{x} \qquad ......(1.17)$$

where $f(t)$ is the damping (resisting) force due to relative velocity $\dot{x}$ between two movable parts and B is the coefficient of viscous friction.

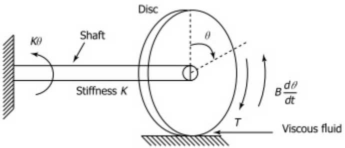
Rotation about a fixed axis takes place in a purely rotational system. The elements are moment of inertia, that is, rotational mass, torsional spring and damping. Moment of inertia is expressed as J. The governing equation relating angular velocity ω with applied torque T, according to Newton's second law of motion, is

$$\begin{aligned} T &= J\omega \\ &= J\frac{d^2\theta}{dt^2} = J\ddot{\theta} \qquad ......(1.18A) \end{aligned}$$

Formulation of equations of translational mechanical system

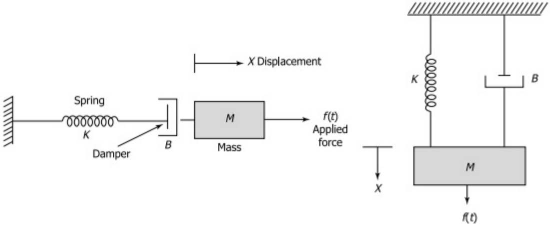
Mechanical elements which experience deformation due to applied torque may be considered rotational spring like a long shaft, helical spring, and so on. The relationship of torque T and angular displacement θ is expressed as

$$T = K\theta \qquad\qquad ......(1.18B)$$



Formulation of equations of translational mechanical system

The mass of the system is M (the unit of M is kg). The displacement of mass due to applied force $f(t)$ (the unit of force is Newton) results in inertia force. This inertia force is the product of the mass and its acceleration. The spring deflection constant is K Newton/metre. The restoring force f_K for the spring is proportional to its displacement. The viscous damping in the system is offered by a dashpot damper. The damping force varies in direct proportion to the velocity. The coefficient of viscous damping is B Newton /rad/sec.



Formulation of equations of translational Mechanical Systems

Application of force $f(t)$ to the mass results in a displacement of x metre. The equation of motion for the system is obtained by applying Newton's Second law of Motion.

$$\text{Inertia force} = \text{mass} \times \text{acceleration} = M\tfrac{d^2x}{dt^2} \qquad \text{......} \ (1.19)$$

$$\text{Applied force} = \text{Inertia force} + \text{Damping force} + \text{Restoring force by the spring} \qquad \text{......} \ (1.20)$$

$$f(t) = M\tfrac{d^2x}{dt^2} + B\tfrac{dx}{dt} + Kx \qquad \text{......}(1.21)$$

Rearranging equation, the following equation is obtained

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t) \qquad \text{......}(1.22)$$

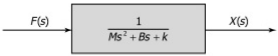
Formulation of equations of translational Mechanical Systems

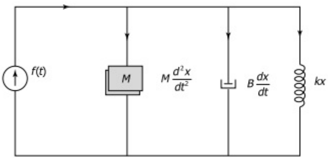
Taking Laplace transform on both sides and assuming all initial conditions to be zero,

$$Ms^2X(s) + BsX(s) + KX(s) = F(s) \qquad ......(1.23)$$

$$\left[Ms^2 + Bs + K\right] X(s) = F(s) \qquad ......(1.23A)$$

$$\text{Transfer function, } G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K} \qquad ......(1.23B)$$





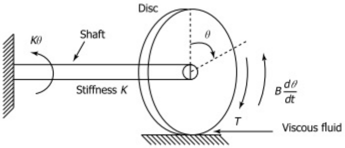
Formulation of equations of Rotational Systems

Let T be the applied torque which tends to rotate the disc. The three basic components of the rotational system are moment of inertia, viscous friction, and spring stiffness (torsional). The system equation can be written using the relation,

Applied Torque = Inertia torque + Damping torque + Angular displacement ...... (1.24)

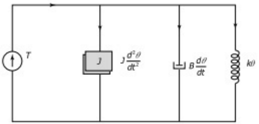
$$\text{Inertia Torque} = \text{Moment of Inertia} \times \text{Angular acceleration} = J\frac{d^2\theta}{dt^2} \quad \text{......} \ (1.25)$$

Thus, $T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta$ ...... (1.26)



Formulation of equations of Rotational Systems

The equivalent circuit diagram of the mechanical system is shown in Fig.



Taking Laplace transform on both sides of torque equation and assuming all initial conditions as zero,

$$T(s) = Js^2\theta(s) + Bs\theta(s) + K\theta(s) \qquad ...... \; (1.27)$$

$$T(s) = \left(Js^2 + Bs + K\right)\theta(s) \qquad ...... \; (1.28)$$

Transfer function, $G(s) = \frac{\theta(s)}{T(s)} = \frac{1}{Js^2+Bs+K}$ ...... (1.29)

Analogies of mechanical and electrical systems

Sometimes mechanical and other systems are converted into electrical analogous systems for the ease of design, modification and analysis. Analogous systems have the same type of differential equations. We will discuss two types of analogies, namely, force-voltage analogy and force-current analogy.

Force-Voltage Analogy

The mathematical representation of mechanical system is

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t) \qquad ......(1.30)$$

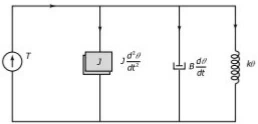
The analogy of this equation can be established by the voltage equation of a RLC electrical circuit. The voltage equation for the circuit is established as follows:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = e(t) \qquad ......(1.31)$$

Electrical circuit for force-voltage analogy
As the current is the rate of flow of electric charge, $i = \frac{dq}{dt}$. Thus, the equation becomes

$$L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad ......(1.32)$$

It is observed that equation (1.29) is analogous to equation (1.32). The three equations, namely, i) the equation for the mechanical translational system represented by mass, spring and damper; ii) the mechanical rotational system represented by moment of i and stiffness; and iii) the electrical system of inductance, resistance, and capacitance placed together as



Force-Voltage Analogy

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t) \qquad ......(1.33)$$

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = T \qquad ......(1.34)$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad ......(1.35)$$

From the above three equations, the analogies between the mechanical translation, mechanical rotational, and electrical system are established and shown in Table below.

Mechanical Translational Systems	Mechanical Rotational Systems	Electrical Systems
Force, f	Torque, T	Voltage, e
Mass, M	Moment of inertia, J	Inductance, L
Viscous friction coefficient, B	Viscous friction coefficient, B	Resistance, R
Spring Stiffness, K	Torsional spring stiffness, K	Reciprocal of capacitance, $1/C$
Displacement, x	Angular displacement, θ	Charge, q
Velocity, $\dot{x}$	Angular velocity, $\dot{\theta}$	Current, i

Force-Voltage Analogy

Mechanical Translational Systems	Mechanical Rotational Systems	Electrical Systems
Force, f	Torque, T	Voltage, e
Mass, M	Moment of inertia, J	Inductance, L
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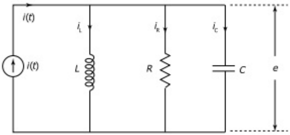
Spring Stiffness, K	Torsional spring stiffness, K	Reciprocal of capacitance, $1/C$
Displacement, x	Angular displacement, θ	Charge, q
Velocity, $\dot{x}$	Angular velocity, $\dot{\theta}$	Current, i

In mechanical system, $f(t)$ is the force applied whereas in electrical system, $e(t)$ is the voltage applied. This is the reason for which the analogy is called force-voltage analogy. In rotational system, force is replaced by torque.



Applying Kirchhoff's current law in the L-R-C parallel circuit shown in Figure, we can write

$$i_L + i_c + i_R = i(t) \qquad ......(1.36)$$

we know that the emf induced in an inductor is $e = L\frac{di}{dt}$; Current through the inductor is, $i_L = \frac{1}{L}\int edt$ and the charge on a capacitor is $q = \int i_c dt$ and

$$q = Ce$$

$$\therefore \int i_c dt = Ce$$

Current through the capacitor i_c is, therefore, Also,

$$i_c = C\frac{de}{dt}$$

$$i_R R = e, i_R = \frac{1}{R}e$$

$$i_L = \frac{1}{L}\int edt; i_R = \frac{1}{R}e; i_c = C\frac{de}{dt}$$

Substituting the values of i_L, i_R and i_c in equation (1.36), we get

$$\frac{1}{L}\int edt + C\frac{de}{dt} + \frac{1}{R}e = i(t) \qquad ......(1.37)$$

In terms of flux linkage, ψ the above equation can be represented by putt in $e = \frac{d\psi}{dt}$ where $\psi = N\varphi$, i.e. the flux linkage Equation (1.37) becomes,

Force–current Analogy

$$\frac{1}{L}\psi + \frac{1}{R}\frac{d\psi}{dt} + C\frac{d^2\psi}{dt^2} = i(t) \quad \left[\because \int edt = \psi\right\} \qquad ......(1.38)$$

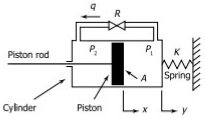
Rearranging, we get

$$C\frac{d^2\psi}{dt^2} + \frac{1}{R}\frac{d\psi}{dt} + \frac{1}{L}\psi = i(t) \qquad ......(1.39)$$

The above equation 1.39 is compared with the equation 1.33 and 1.34 The analogies between the mechanical translation, mechanical rotational, and electrical system are established and shown in Table below.

Mechanical Translational Systems	**Mechanical Rotational Systems**	**Electrical Systems**
Force, f	Torque, T	Current, i
Mass, M	Moment of inertia, J	Capacitance, C
Viscous friction coefficient, B	Viscous friction coefficient, B	Reciprocal of resistance, $1/R$
Spring stiffness, K	Torsional spring stiffness, K	Reciprocal of inductance, $1/L$
Displacement, x	Angular displacement, θ	Flux linkage, ψ
Velocity, $\dot{x}$	Angular velocity, $\dot{\theta}$	Voltage, e

Let us determine the transfer function of a typical hydraulic system. A hydraulic system works due to liquid pressure difference.
A dashpot (or damper) is shown in Figure. Whenever a step displacement, x is applied to the piston, a corresponding displacement y becomes momentarily equal to x. The applied force will make oil to flow through the restriction R and the cylinder will return to its original position.



Assuming inertia force to be a negligible, the forces balancing are represented as $A\,(p_1 - p_2) = ky \ldots (2.19)$ where A is the piston area, k is the spring constant and p_1, p_2 are the oil pressure existing on the right and left side of the piston.

Hydraulic system

The flow rate q through the restriction with resistance R is given by

$$q = \frac{p_1 - p_2}{R}$$

The oil is assumed to be incompressible (so oil density ρ = constant). As the mass of oil flow through the restriction in time dt must balance with the change in mass of the left side of the piston, we have

$$qdt = A\rho(dx - dy)$$

Rearranging and using equations (2.19) and (2.20), we obtain or,

$$\frac{dx}{dt} - \frac{dy}{dt} = \frac{q}{A\rho} = \frac{p_1 - p_2}{RA\rho} = \frac{ky}{RA^2\rho}$$
$$\frac{dx}{dt} = \frac{dy}{dt} + \frac{ky}{RA^2\rho}$$

Taking Laplace transform with zero initial conditions, we get

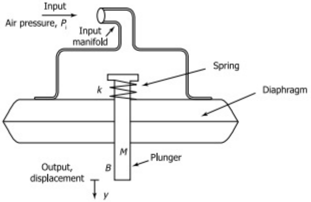
$$sX(s) = sY(s) + \frac{k}{RA^2\rho}Y(s)$$

So, the transfer function of the system becomes

$$G(s) = \frac{Y(s)}{X(s)} = \frac{s}{s + \frac{k}{RA^2\rho}} = \frac{s}{s + \frac{1}{\tau}} = \frac{\tau s}{1 + \tau s}$$

where $\tau = \frac{RA^2\rho}{k}, \tau$ is called the time constant.

A pneumatic system works due to the pressure difference of air or any other gas. We shall consider a simple pneumatic system with an actuating valve as shown in Figure.



Air at a pressure, p_i is injected through the input manifold. The plunger arrangement has a mass M, B and K being the coefficients of viscous friction and spring constant respectively. If A is the area of diaphragm, then the force exerted on the system will be Ap_r Force $F =$ Pressure $\times$ Area that is,

$$F = p_i A = A p_i$$

$$F = M\frac{d^2y}{dt^2} + B\frac{dy}{dt} + ky$$

Again, $\therefore$

$$M\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Ky = AP_i$$

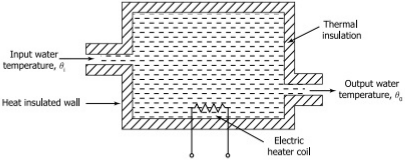
Taking Laplace transform of both sides and assuming initial conditions to be zero, we get

$$Ms^2Y(s) + BsY(s) + KY(s) = AP_i(s)$$

The transfer function is

$$\text{Transfer function, } G(s) = \frac{Y(s)}{P_i(s)} = \frac{A}{Ms^2 + Bs + K}$$

Figure shows a water heating system (a thermal system) where input water temperature is θ_i and output water temperature is θ_0



The following assumptions are made to analyse the thermal system and determine its transfer function.

i) The temperature of the medium is uniform

ii) The tank is insulated from the surrounding atmosphere.

Let the steady state temperature of inflowing water is θ_i and that of outflowing water is θ_0. The steady state heat input rate from the heater is Q. Let the water flow rate be constant. Any increase in heat input rate, ΔH will be balanced by increase in heat outflow rate, ΔH_1 and heat storage rate in the tank, ΔH_2.

Using heat balance equation, we can write

$$\Delta H = \Delta H_1 + \Delta H_2$$
$$\Delta H_1 = QS\Delta\theta_0$$

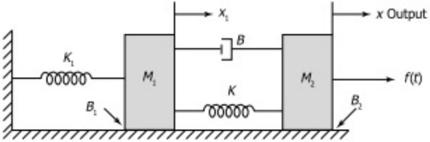
where Q is water flow rate and S is the specific heat

$$\Delta H_1 = \frac{\Delta\theta_0}{R}$$

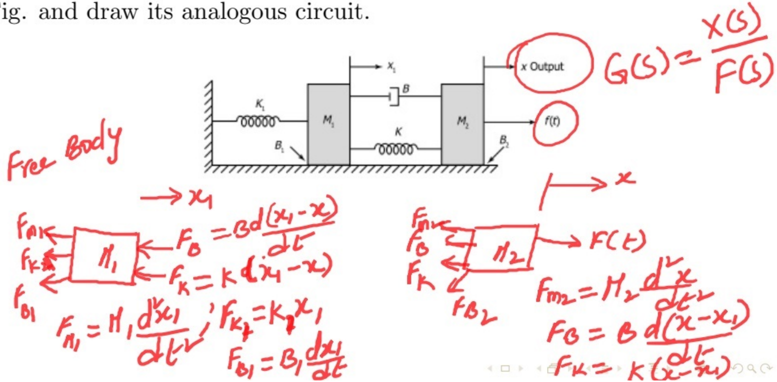
where $R = \frac{1}{QS}$, is called the thermal resistance Heat storage rate in the tank,

$$\Delta H_2 = MS\frac{d}{dt}(\Delta\theta_0)$$
$$= C\frac{d}{dt}(\Delta\theta_0)$$

where M is the mass of water in the tank; $C = MS$, is the thermal capacitance of water and $\frac{d(\Delta\theta_0)}{dt}$ is the rate of rise of temperature of water in the tank. Now from equations (2.23), (2.24) and (2.25) we can write, Prob1: Obtain the transfer function of the mechanical system shown in Fig. and draw its analogous circuit.



Prob1: Obtain the transfer function of the mechanical system shown in Fig. and draw its analogous circuit.



Problem1

$$F_{B_2} = B_2 \frac{dx}{dt}$$

Force balance eqt on Mass element 1, according to New 2nd Law

$$F_{M_1} + F_{K_1} + F_{B_1} + F_B + F_K = 0 \quad \text{--- (1)}$$

|||ly apply New 2nd Law on Mass element 2, we get

$$F_{M_2} + F_B + F_K + F_{B_2} = f(t) \quad \text{--- (2)}$$

Problem1

Substitute the individual forces from the free body diagram in eq^n ① & ② we get,

$$M_1\frac{d^2x_1}{dt}+B_1\frac{dx_1}{dt}+K_1x_1+B\frac{d(x_1-x)}{dt^2}+K(x_1-x)=0 \quad —(3)$$

$$M_2\frac{d^2x}{dt^2}+B_2\frac{dx}{dt}+B\frac{d(x-x_1)}{dt}+K(x-x_1)=f(t) \quad —(4)$$

Apply LT's for eq^n ③, we get

$$M_1S^2X_1(S)+B_1SX_1(S)+K_1X_1(S)+B(SX_1(S)-SX(S))+K(X_1(S)-X(S))=0 \quad —(5)$$

Apply LT's for eqn (4), we get

$$M_2 s^2 X(s) + B_2 s X(s) + B(sX(s) - sX_1(s)) + K(X(s) - X_1(s)) = F(s) \quad \text{---(6)}$$

from eqn (5), write down $X_1(s)$ interms of $X(s)$

$$(M_1 s^2 + B_1 s + K_1 + Bs + K)X_1(s) - (Bs + K)X(s) = 0$$

$$X_1(s) = \frac{(Bs+K)}{[M_1 s^2 + (B_1+B)s + (K_1+K)]} X(s) \quad \text{---(7)}$$

Substitute $X_1(s)$ in eqn (6)

$$M_2 s^2 X(s) + B_2 sX(s) + BsX(s) - Bs\left\{\frac{(Bs+K)}{[M_1 s^2 + (B_1+B)s + (K_1+K)]}X(s)\right.$$

Problem1

$$M_2 s^2 X(s) + B_2 s X(s) + B s X(s) - B s \frac{(Bs+K)}{[M_1 s^2 + (B_1+B)s + (K_1+K)]} X(s)$$

$$+ K X(s) - K \frac{(Bs+K) X(s)}{[M_1 s^2 + (B_1+B)s + (K_1+K)]} = F(s)$$

$$\left[M_2 s^2 + (B_2+B)s + K - \frac{(Bs+K)(Bs+K)}{[M_1 s^2 + (B_1+B)s + (K_1+K)]}\right](X(s)) = F(s)$$

Transfer function

$$G(s) = \frac{X(s)}{F(s)} = \frac{[M_1 s^2 + (B_1+B)s + (K_1+K)]}{[(M_2 s^2 + (B_2+B)s + K)(M_1 s^2 + (B_1+B)s + (K_1+K)) - (Bs+K)^2]}$$

$$M_1 M_2 s^4 + M_1 (B_2+B) s^3 + M_2 (B_1+B) s^3 + M_1 K s^2 + M_2 (K_1+K) s^2 + B_1 B s^2 + B_2 B s^2 + \dots$$

Problem1

$$+ B_1 B s^2 + B s^2 + (B_2 + B)(K_1 + K)s + (B_1 + B)Ks + K_1 + K^2 - B^2 s^2 - 2BKs - K^2$$

$$B_2 K_1 s + B K_1 s + B_2 K s + B K s$$

$$G(s) = \frac{\left[M_1 s^2 + (B_1 + B)s + (K_1 + K)\right]}{\left[M_1 M_2 s^4 + M_1(B_2 + B)s^3 + M_2(B_1 + B)s^3 + M_1 K s^2 + M_2(K_1 + K)s^2 + B_1 B_2 s^2 + B_2 B s^2 + B_1 B s^2 + B_2 K_1 s + B K_1 s + B_2 K s + B_1 K s + K_1 K\right]}$$

$$G(s) = \frac{\left[M_1 s^2 + (B_1 + B)s + (K_1 + K)\right]}{\left[M_1 M_2 s^4 + (M_1 B_2 + M_1 B + M_2 B_1 + M_2 B)s^3 + (M_1 K + M_2 K_1 + M_2 K + B_1 B_2 + B_2 B + B_1 B)s^2 + (B_2 K_1 + B K_1 + B_2 K + B_1 K)s + K_1 K\right]} \quad -(8)$$

Problem1

$$M_1 \frac{d^2 x_1}{dt} + B_1 \frac{dx_1}{dt} + K_1 x_1 + B \frac{d(x_1 - x)}{dt^2} + K(x_1 - x) = 0 \quad \text{— (3)}$$

Force – voltage

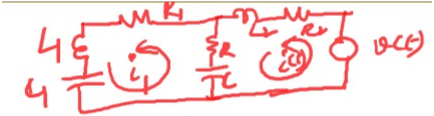
$$L_1 \frac{d^2 q_1}{dt^2} + R_1 \frac{dq_1}{dt} + \frac{1}{C_1} q_1 + R \frac{d(q_1 - q)}{dt} + \frac{1}{C}(q_1 - q) = 0$$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 \, dt + R(i_1 - i) + \frac{1}{C}\int (i_1 - i)\, dt = 0 \quad \text{— (9)}$$

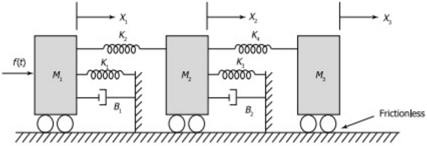
$$M_2 \frac{d^2 x}{dt^2} + B_2 \frac{dx}{dt} + B \frac{d(x - x_1)}{dt} + K(x - x_1) = f(t) \quad \text{— (4)}$$

$$L_2 \frac{di}{dt} + R_2 i + R(i - i_1) + \frac{1}{C}\int (i - i_1)\, dt = v(t) \quad \text{— (10)}$$

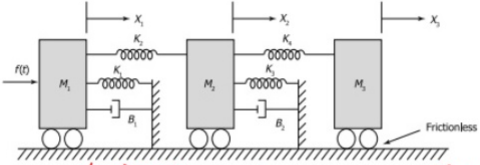
Problem1

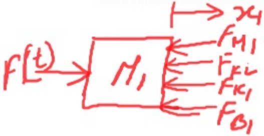


Prob2: Draw the mechanical network for the system shown in Fig and draw its analogous circuit.



Problem2





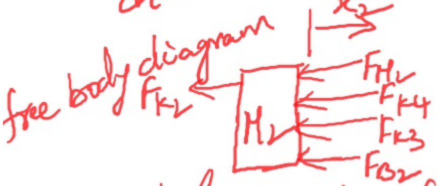
$F_{M_1} = M_1 \frac{d^2x_1}{dt^2}$ $F_{K_2} = K_2(x_1 - x_2)$

$F_{K_1} = K_1 x_1$ $F_{B_1} = B_1 \frac{dx_1}{dt}$

Force balance eqn for M_1

$$F(t) = F_{M_1} + F_{K_2} + F_{K_1} + F_{B_1}$$

$$M_1\frac{d^2x_1}{dt^2} + K_2(x_1 - x_2) + K_1x_1 + B_1\frac{dx_1}{dt} = f(t) \quad \text{--(1)}$$



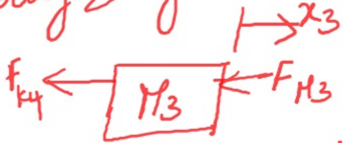
Force balance eqn for M_2 is

$$F_{K_2} + F_{M_2} + F_{K_4} + F_{K_3} + F_{B_2} = 0$$

Problem2

$$K_2(x_2-x_1)+M_2\frac{d^2x_2}{dt^2}+K_4(x_2-x_3)+K_3x_2+B_2\frac{dx_2}{dt}=0 \quad \text{—(2)}$$

free body Diagram

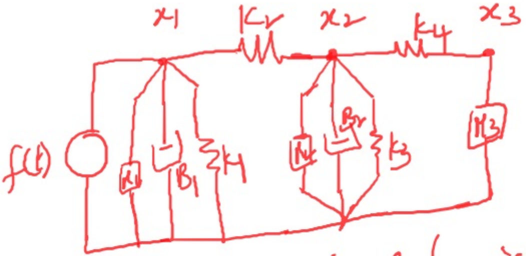


force balance eqn for M_3 is

$$F_{K4}+F_{M3}=0$$

$$K_4(x_3-x_2)+M_3\frac{d^2x_3}{dt^2}=0 \quad \text{—(3)}$$

Problem2



Equivalent ckt for Mechanical System

Force current analogous

$M_1 \to C_1$; $B_1 \to 1/R_1$; $K_1 \to \frac{1}{L_1}$; $K_2 \to \frac{1}{L_2}$

$M_2 \to C_2$; $B_2 \to 1/R_2$; $K_3 \to \frac{1}{L_3}$

Problem2

$K_4 \Rightarrow \frac{1}{L_4}$; $M_3 \rightarrow C_3$

$$M_1 \frac{d^2 x_1}{dt^2} + K_2(x_1 - x_2) + K_1 x_1 + B_1 \frac{dx_1}{dt} = f(t) \quad \text{---(1)}$$

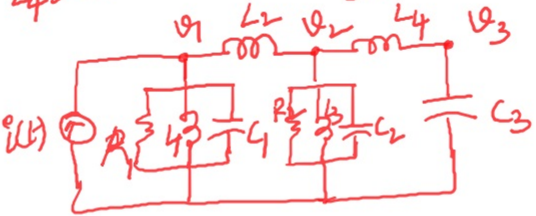
$$C_1 \frac{d^2 \psi_1}{dt^2} + \frac{1}{L_2}(\psi_1 - \psi_2) + \frac{1}{L_1}\psi_1 + \frac{1}{R_1}\frac{d\psi_1}{dt} = i(t)$$

$$C_1 \frac{dv_1}{dt} + \frac{1}{L_2}\left(\int (v_1 - v_2)\,dt\right) + \frac{1}{L_2}\int v_1\,dt + \frac{1}{R} v_1 = i(t) \quad \text{---(4)}$$

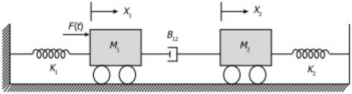
Problem2

$$\frac{1}{L_2}\int(v_2-v_1)\,dt + C_2\frac{dv_2}{dt} + \frac{1}{L_4}\int(v_2-v_3)\,dt + \frac{1}{L_3}\int v_2\,dt + \frac{1}{R_2}v_2 = 0 \quad \text{---(5)}$$

$$\frac{1}{L_4}\int(v_3-v_2)\,dt + C_3\frac{dv_3}{dt} = 0 \quad \text{---(6)}$$



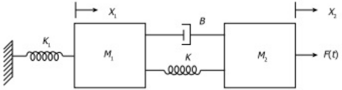
Prob3: Draw the mechanical network for the system shown in Fig and draw its analogous circuit.



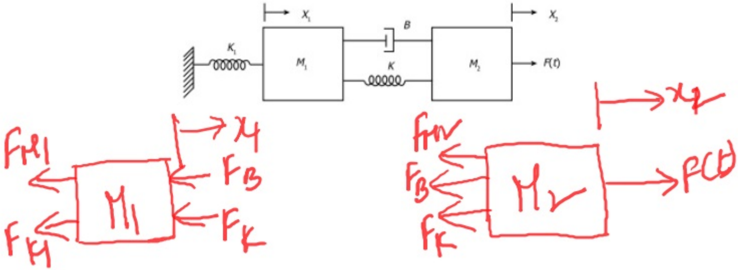
Problem3

Home Work

Prob4: Using force–voltage analogy and force–current analogy draw the equivalent analogous system for the system shown in Figure.



Problem4



Problem4

force balance eqn for M_1

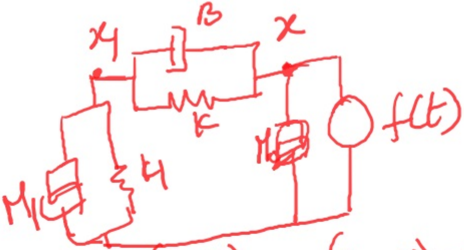
$$F_{M_1}+F_{K_4}+F_B+F_K=0 \quad \text{——(1)}$$

$$M_1\frac{d^2x_4}{dt^2}+K_4x_4+B\frac{d(x_4-x)}{dt}+K(x_4-x)=0 \quad \text{——(2)}$$

force balance eqn for M_2

$$F_{M_2}+F_B+F_K=f(t) \quad \text{——(3)}$$

$$M_2\frac{d^2x}{dt^2}+B\frac{d(x-x_4)}{dt}+K(x-x_4)=f(t) \quad \text{——(4)}$$



$$M_1\frac{d^2x_1}{dt^2}+K_1x_1+B\frac{d(x_1-x)}{dt}+K(x_1-x)=0 \quad -(4)$$

F–V

$$L_1\frac{d^2q_1}{dt^2}+\frac{1}{C_1}q_1+R\frac{d(q_1-q)}{dt}+\frac{1}{C}(q_1-q)=0 \quad -(5)$$

Problem4

$$L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1\,dt + R(i_1 - i) + \frac{1}{C}\int (i_1 - i)\,dt = 0 \quad \text{---(6)}$$

$$M_2 \frac{d^2x}{dt^2} + B\frac{d(x - x_1)}{dt} + K(x - x_1) = f(t) \quad \text{---(4)}$$

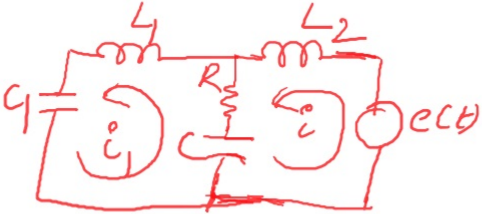
$$L_2 \frac{d^2q}{dt^2} + R\frac{d(q - q_1)}{dt} + \frac{1}{C}(q - q_1) = e(t) \quad \text{---(7)}$$

$$L_2 \frac{di}{dt} + R(i - i_1) + \frac{1}{C}\int (i - i_1)\,dt = e(t) \quad \text{---(8)}$$

Problem4

F-V equivalent ckt

Take eqt (6) & (8), we draw Analagous ckt (F-V)



Problem4

F-I

$$v_c = \frac{1}{C}\int i\,dt \Rightarrow i = C\frac{dv}{dt}$$

$$M_1\frac{d^2x_1}{dt^2} + K_1x_1 + B\frac{d(x_1-x)}{dt} + K(x_1-x) = 0 \quad \text{—(8)}$$

$$C_1\frac{d^2\psi_1}{dt^2} + \frac{1}{L_1}\psi_1 + \frac{1}{R}\frac{d(\psi_1-\psi)}{dt} + \frac{1}{L}(\psi_1-\psi) = 0 \quad \text{—(9)}$$

$$C_1\frac{dv_1}{dt} + \frac{1}{L_1}\int v_1\,dt + \frac{1}{R}(v_1-v) + \frac{1}{L}\int(v_1-v)\,dt = 0 \quad \text{—(10)}$$

Problem4

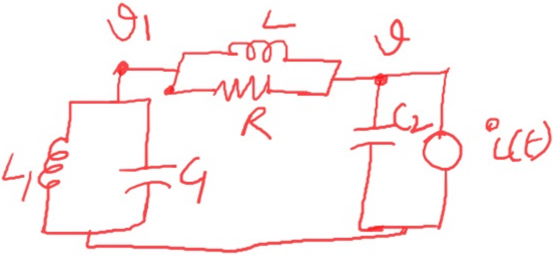
$$M_2\frac{d^2x}{dt^2}+B\frac{d(x-x_4)}{dt}+K(x-x_4)=f(t) \quad \text{---(4)}$$

$$C_2\frac{d^2\varphi}{dt^2}+\frac{1}{R}\left(\frac{d(\varphi-\varphi_1)}{dt}\right)+\frac{1}{L}(\varphi-\varphi_1)=i(t) \quad \text{---(1)}$$

$$C_2\frac{dv}{dt}+\frac{1}{R}(v-v_1)+\frac{1}{L}\int(v-v_1)\,dt=i(t) \quad \text{---(2)}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

Problem4



F–I analogous circuit

Ture/False Questions

Q.1 State whether the following statements are True or False.

i Feedback control systems are also referred to as closed-loop control systems

ii A washing machine where the washing is done on a time basis through a timer is an example of closed-loop control system

iii In an open-loop the control operation is independent of the output

iv In an error detector the feedback signal is added with the reference signal to obtain error signal

v A refrigerator is an example of closed-loop control system

vi Performance of an open-loop system depends on the settings of the system components

vii Fixed-time traffi c light control system is an example of closed-loop control system

Ture/False Questions

viii In a linear system the response produced by simultaneous action of two different forcing functions is the difference of individual responses

ix In stochastic control system the response is not predictable and repeatable

x Servomechanism is an automatic control system

xi An open-loop system is a system without feedback

xii A control system is an interconnection of components forming a system configuration that provides a desired system response

xiii A closed-loop control system uses a measurement of the output and compares it with the desired input (reference or command)

xiv A negative feedback control system is the one where the output signal is fed back so that it is added to the input signal

Fill in the blanks

Q.1 Complete the following sentences.

i) The transfer function of a linear system is defined as the ratio of the __________ of the output variable to the __________ of the input variable

ii) Transfer function of an integrating circuit is $G(s) =$ ________________

iii) Transfer function of a differentiating circuit is $G(s) =$ ________________

iv) First step in determining the transfer function of a control system is to formulate the __________ for the system

v) If the transfer function of a system is known, we can determine the behaviour of the system for _________ to understand the nature of system

vi) A system is described by the differential equation $4\ddot{y} + 2\dot{y} + y = 5\dddot{x} + 3\ddot{x} + 2\dot{x} + x$, where x is the system output and y is the system input. The transfer function of the system is _______________

vii) Transfer function of a mechanical system having mass, spring and damper is calculated as $G(s) =$ _______________

Q.2 A closed-loop control system can be defined as

a system that directly generates the output in response to an input signal

b system with a measurement of the output signal and a comparison with the desired output to generate an error signal that is applied to the actuator

c an interconnection of components forming a system configuration that will provide a desired response, the output having no effect upon the signal to the process

d a system that utilises a device to control the process without using feedback

Q.3 In a multivariable control system there is

a more than one input variable but one unique output

b one input variable but variable outputs

c more than one input variable or more than one output variable

d more than one input variable and more than one output variable

Q.4 A negative feedback control system is the one where

a the output signal is fed back as it subtracts from the input signal to provide desired response

b the output signal is fed back so that it adds to the input signal to reduce the output

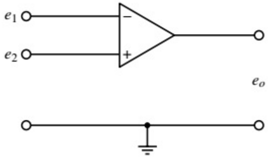
c the input signal is reduced so as to reduce the output when it is more than the desired value

d control of the process is achieved without using any feedback

Operational amplifiers, often called op amps, are frequently used to amplify signals in sensor circuits. Op amps are also frequently used in filters used for compensation purposes. Figure shows an op amp. It is a common practice to choose the ground as 0 volt and measure the input voltages e1 and e2 relative to the ground. The input e1 to the minus terminal of the amplifier is inverted, and the input e2 to the plus terminal is not inverted.The total input to the amplifier thus becomes e2-e1.

$$e_o = K\left(e_2 - e_1\right) = -K\left(e_1 - e_2\right)$$

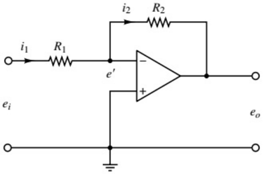
where the inputs e1 and e2 may be dc or ac signals and K is the differential gain (voltage gain).



Note that the op amp amplifies the difference in voltages e1 and e2. Such an amplifier is commonly called a differential amplifier. Since the gain of the op amp is very high, it is necessary to have a negative feedback from the output to the input to make the amplifier stable. (The feedback is made from the output to the inverted input so that the feedback is a negative feedback.) In the ideal op amp, no current flows into the input terminals, and the output voltage is not affected by the load connected to the output terminal. In other words, the input impedance is infinity and the output impedance is zero. In an actual op amp, a very small (almost negligible) current flows into an input terminal and the output cannot be loaded too much. In our analysis here, we make the assumption that the op amps are ideal.

Inverting Amplifier

Consider the operational-amplifier circuit shown in Figure.



Let us obtain the output voltage eo. The equation for this circuit can be obtained as follows: Define

$$i_1 = \frac{e_i - e'}{R_1}, \quad i_2 = \frac{e' - e_o}{R_2}$$

Since only a negligible current flows into the amplifier, the current i_1 must be equal to current i_2. Thus

$$\frac{e_i - e'}{R_1} = \frac{e' - e_o}{R_2}$$

Since $K(0 - e') = e_0$ and $K \geqslant 1, e'$ must be almost zero, or $e' \doteq 0$. Hence we have

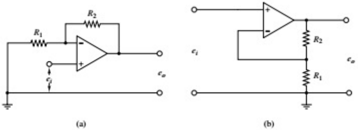
$$\frac{e_i}{R_1} = \frac{-e_o}{R_2}$$

or

$$e_e = -\frac{R_2}{R_1} e_i$$

Thus the circuit shown is an inverting amplifier. If $R_1 = R_2$, then the op-amp circuit shown acts as a sign inverter.

Figure a shows a noninverting amplifier. A circuit equivalent to this one is shown in Figure b. For the circuit of Figure b, we have



$$e_o = K\left(e_i - \frac{R_1}{R_1 + R_2} e_e\right)$$

where K is the differential gain of the amplifier. From this last equation, we get

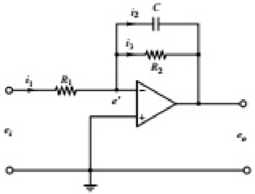
$$e_i = \left(\frac{R_1}{R_1 + R_2} + \frac{1}{K}\right) e_o$$

Since $K \geqslant 1$, if $R_1/\left(R_1 + R_2\right) \geqslant 1/K$, then

$$e_e = \left(1 + \frac{R_2}{R_1}\right) e_i$$

This equation gives the output voltage e_o. Since e_o and e_i have the

Figure shows an electrical circuit involving an operational amplifier. Obtain the transfer function.



Problem

Problem

- Any control system will have a number of control components. A control system can be represented in block diagram form.
- For making a block diagram, first the transfer function of the system components are determined. They are then showed in respective blocks. These blocks are connected by arrows indicating the direction of flow of signals in the control system whose block diagram is being represented. The signals can pass only in the direction of arrows represented in the diagram.
- The arrow head pointing towards a particular block indicates the input to the system component and the arrow head leading away from the block indicates the output. All the arrows in a block diagram are referred to as signals.

ADVANTAGES OF BLOCK DIAGRAM REPRESENTATION

The following are the main advantages of block diagram representation of control systems.

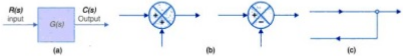
- The overall block diagram of a system can be easily drawn by connecting the blocks according to the signal flow. It is also possible to evaluate the contribution of each of the components towards the overall performance of the control system.
- Block diagram helps in understanding the functional operation of the system more readily than examination of the actual control system physically. It may be noted that a block diagram drawn for a system is not unique, that is, there may be alternative ways of representation of a system in block diagram form.

BLOCK DIAGRAM REPRESENTATION

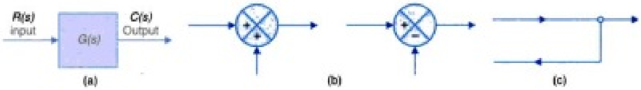
As introduced earlier, the input-output behaviour of a linear system or element of a linear system is given by its transfer function

$$G(s) = \frac{C(s)}{R(s)}$$

where $R(s)$ = Laplace transformation of the input variable; and $C(s)$ = Laplace transform of the output variable.
A convenient graphical representation of this behaviour is the block diagram as shown in Fig (a) wherein the signal into the block represents the input $R(s)$ and the signal out of the block represents the output $C(s)$, while the block itself stands for the transfer function $G(s)$. The flow of information (signal) is unidirectional from the input to the output with the output being equal to the input multiplied by the transfer function of the block.

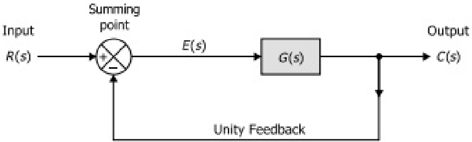


A complex system comprising of several non-loading elements is represented by the interconnection of the blocks for individual elements. The blocks are connected by lines with arrows indicating the unidirectional flow of information from the output of one block to the input of the other. In addition to this, summing or differencing of signals is indicated by the symbols shown in Fig (b), while the take-off point of a signal is represented by Fig(c).



BLOCK DIAGRAM OF A CLOSED-LOOP SYSTEM AND ITS TRANSFER FUNCTION

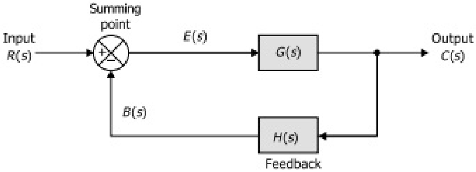
The basic block diagram representation of a unity feedback control system has been shown



When the output is fed back to the summing point for comparison with the input to create an error signal, it is necessary to convert the form of the output signal to that of the input signal. The quantities being added or subtracted at the summing point should have the same dimensions and therefore, should have the same units. The conversion of a fraction of output signal on feedback path is done by the feedback element whose transfer function is H(s). The feedback element modifies the output before it is compared with the input signal.

BLOCK DIAGRAM OF A CLOSED-LOOP SYSTEM AND ITS TRANSFER FUNCTION

Fig. shows the standard form of representation of a feedback control system in block diagram form. Here a fraction of output B(s) = C(s) H(s) is brought to the summing point thereby producing an error signal. This block diagram is also called the canonical form of representation of a control system. G(s) is the system transfer function.



Referring to Fig we can write, $E(s) = R(s) - B(s)$
$B(s) = H(s)C(s)$ and $\quad C(s) = E(s)G(s)$
$\therefore \quad E(s) = R(s) - H(s)C(s)$ or, $\quad R(s) = E(s) + H(s)C(s)$

BLOCK DIAGRAM OF A CLOSED-LOOP SYSTEM AND ITS TRANSFER FUNCTION

$R(s) = E(s) + H(s)E(s)G(s)$ Therefore, the closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{E(s)G(s)}{E(s) + H(s)G(s)E(s)}$$
$$= \frac{G(s)}{1 + G(s)H(s)}$$

Transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The above function is the closed loop transfer function for negative fecd-back.

BLOCK DIAGRAM OF A CLOSED-LOOP SYSTEM AND ITS TRANSFER FUNCTION

The transfer function for a closed loop system with positive feedback is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

This is because, the error signal for positive feedback will be, $E(s) = R(s) + B(s)$. For a unity fecdback control system, $H(s) = 1$. The transfer function for a unity focdback system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)}$$

where plus sign in the denominator stands for negative feedback.

BLOCK DIAGRAM REPRESENTATION OF COMPONENTS OF A SERVOMECHANISM

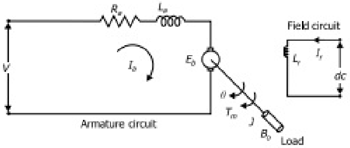
A servomechanism, also called a position control system, is a feedback control system and consists of a mechanism in which the output of the system may be some mechanical position, velocity or acceleration. Servomechanism and position control systems are synonymous. In almost all servomechanism applications d.c. motor drives and gear mechanism are used.

In servo applications, a DC motor is required to produce rapid acceleration from stand still (position of rest). Therefore, the physical requirements of such a motor are low inertia and high starting torque. Low inertia is attained with reduced armature diameter.

In control systems, DC motors are used in two different control modes: a) Armature control mode with constant field current; and b) Field control mode with fixed armature current.

Transfer function of the DC motor

The DC motor is a power actuator device that delivers energy to a load, as shown in Figure (a). The DC motor converts direct current (DC) electrical energy into rotational mechanical energy. A major fraction of the torque generated in the rotor (armature) of the motor is available to drive an external load. Because of features such as high torque, speed controllability over a wide range, portability, well-behaved speed-torque characteristics, and adaptability to various types of control methods, DC motors are widely used in numerous control applications, including robotic manipulators, tape transport mechanisms, disk drives, machine tools, and servo valve actuators.



Transfer function of the DC motor

The transfer function of the DC motor will be developed for a linear approximation to an actual motor, and second-order effects, such as hysteresis and the voltage drop across the brushes, will be neglected. The input voltage may be applied to the field or armature terminals. The air-gap flux $\phi(t)$ of the motor is proportional to the field current, provided the field is unsaturated, so that

$$\phi(t) = K_f i_f(t)$$

The torque developed by the motor is assumed to be related linearly to $\phi(t)$ and the armature current as follows:

$$T_m(t) = K_1 \phi(t) i_a(t) = K_1 K_f i_f(t) i_a(t)$$

Transfer function of the DC motor

First, we shall consider the field current controlled motor, which provides a substantial power amplification. Then we have, in Laplace transform notation,

$$T_m(s) = (K_1 K_f I_a)\, I_f(s) = K_m I_f(s)$$

where $i_a = I_a$ is a constant armature current, and K_m is defined as the motor constant. The field current is related to the field voltage as

$$V_f(s) = (R_f + L_f s)\, I_f(s)$$

The motor torque $T_m(s)$ is equal to the torque delivered to the load. This relation may be expressed as

$$T_m(s) = T_L(s) + T_d(s)$$

where $T_L(s)$ is the load torque and $T_d(s)$ is the disturbance torque, which is often negligible.

Transfer function of the DC motor

However, the disturbance torque often must be considered in systems subjected to external forces such as antenna wind-gust forces. The load torque for rotating inertia, as shown in Figure, is written as

$$T_L(s) = Js^2\theta(s) + bs\theta(s)$$

$$T_L(s) = T_m(s) - T_d(s)$$
$$T_m(s) = K_m I_f(s)$$
$$I_f(s) = \frac{V_f(s)}{R_f + L_f s}$$

Therefore, the transfer function of the motor-load combination, with $T_d(s) = 0$, is

$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js+b)(L_f s + R_f)} = \frac{K_m/(JL_f)}{s(s+b/J)(s+R_f/L_f)}$$

Transfer function of the DC motor

The block diagram model of the field-controlled DC motor is shown in Figure. Alternatively, the transfer function may be written in terms of the time constants of the motor as

$$\frac{\theta(s)}{V_f(s)} = G(s) = \frac{K_m/\left(bR_f\right)}{s\left(\tau_f s+1\right)\left(\tau_L s+1\right)}$$

where $\tau_f = L_f/R_f$ and $\tau_L = J/b$. Typically, one finds that $\tau_L > \tau_f$ and often the field time constant may be neglected.

The armature-controlled DC motor uses the armature current i_a as the control variable. The stator field can be established by a field coil and current or a permanent magnet. When a constant field current is established in a field coil, the motor torque is

$$T_m(s) = \left(K_1 K_f I_f\right) I_a(s) = K_m I_a(s)$$

When a permanent magnet is used, we have

$$T_m(s) = K_m I_a(s)$$

where K_m is a function of the permeability of the magnetic material.

Transfer function of the DC motor

The armature current is related to the input voltage applied to the armature by

$$V_a(s) = (R_a + L_a s)\, I_a(s) + V_b(s)$$

where $V_b(s)$ is the back electromotive-force voltage proportional to the motor speed. Therefore, we have

$$V_b(s) = K_b \omega(s)$$

where $\omega(s) = s\theta(s)$ is the transform of the angular speed and the armature current is

$$I_a(s) = \frac{V_a(s) - K_b \omega(s)}{R_a + L_a s}$$

$$T_L(s) = Js^2\theta(s) + bs\theta(s) = T_m(s) - T_d(s)$$

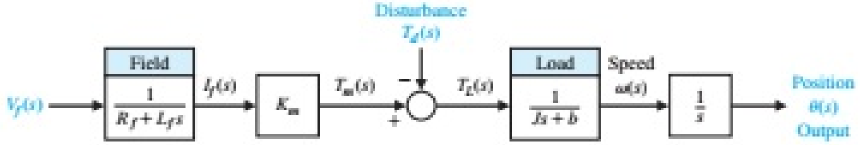
$$G(s)=\frac{\theta(s)}{V_a(s)}=\frac{K_m}{s\left[\left(R_{\mathrm{a}}+L_a s\right)(J s+b)+K_b K_{mn}\right]}$$
$$=\frac{K_m}{s\left(s^2+2\zeta\omega_{nr} s+\omega_n^2\right)}$$

However, for many DC motors, the time constant of the armature, $\tau_a=L_a/R_a$, is negligible; therefore,

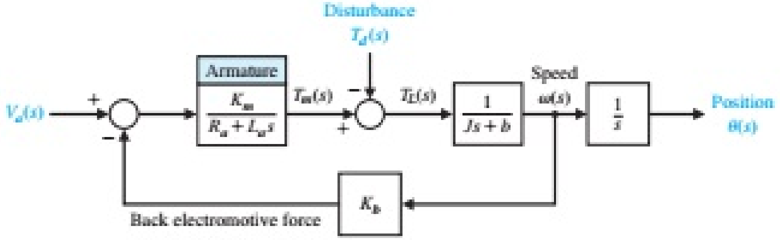
$$G(s)=\frac{\theta(s)}{V_a(s)}=\frac{K_m}{s\left[R_{\mathrm{a}}(J s+b)+K_b K_m\right]}=\frac{K_m/\left(R_d b+K_b K_m\right)}{s\left(\tau_1 s+1\right)}$$

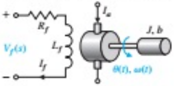
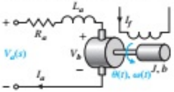
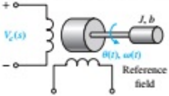
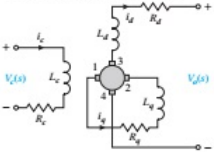
where the equivalent time constant $\tau_1=R_a J/\left(R_a b+K_b K_m\right)$.

Block Diagram of the field controlled DC motor



Block Diagram of the armature controlled DC motor



Element or System	G(s)
DC motor, field-controlled, rotational actuator 	$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$
DC motor, armature-controlled, rotational actuator 	$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$
AC motor, two-phase control field, rotational actuator 	$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$ $\tau = J/(b - m)$ m = slope of linearized torque-speed curve (normally negative)
Rotary Amplifier (Amplidyne) 	$\frac{V_o(s)}{V_c(s)} = \frac{K/(R_c R_q)}{(s\tau_c + 1)(s\tau_q + 1)}$ $\tau_c = L_c/R_c, \quad \tau_q = L_q/R_q$ for the unloaded case, $i_d \approx 0$, $\tau_c \approx \tau_q$, $0.05\ \text{s} < \tau_c < 0.5\ \text{s}$ $V_q, V_{34} = V_d$

Reduction of Block diagram and it's Rules

Complex systems are represented by the interconnection of many subsystems.

In order to analyze our system, we want to represent multiple subsystems as a single transfer function.

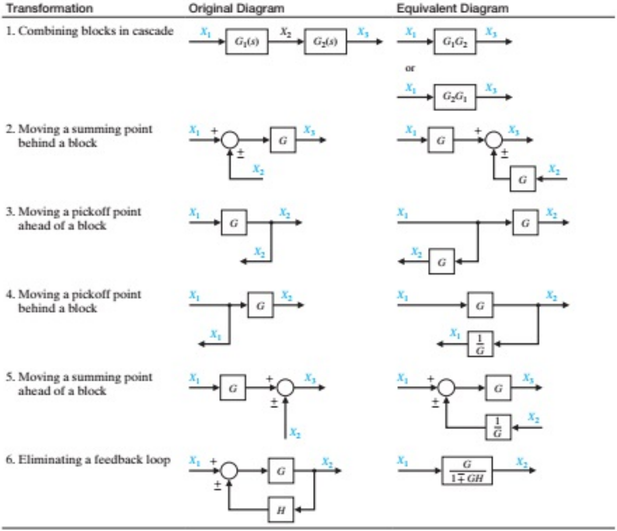
There are three topologies that can be used to reduce a complicated system to a single block.

Cascade form

Parallel form

Feedback form

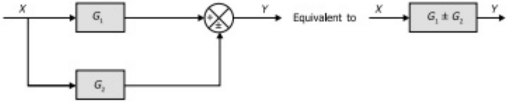
Reduction of Block diagram and it’s Rules



a) When two or more blocks are connected in series, we are to multiply the transfer functions and put as one block as shown in Fig.

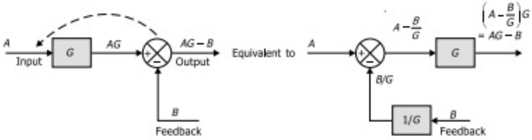
Equivalent of series connected blocks

b) When two blocks are connected in parallel, the transfer functions are to be added as shown in Fig.



Equivalent of parallel connected blocks

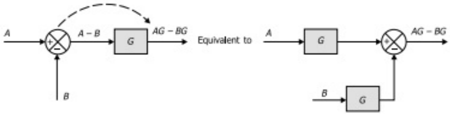
c) When shifting the summing point prior to, i.e. before a block,



Shifting a summing point prior to a block

Note that in this shifting the original input, output and feedback quantities, i.e. A, $AG - B$, and B respectively, must not change. Fig. shows shifting of the summing point before block G with the help of a dotted line.

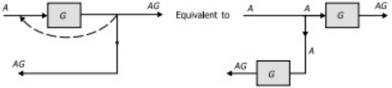
d) Shifting the summing point beyond, i.e. after a block.



Shifting a summing point after a block

As shown in Fig. , to keep the output signal unaltered, a block G has been placed in the feedback path.

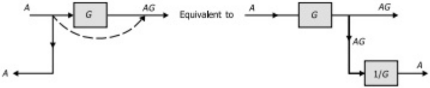
e) Moving a take-off point from after a block to before a block.



Moving a take-off point to before a block

As shown in Fig. , a block G has to be introduced in the feedback path to keep feedback signal as AG unaltered.

f) Moving a take-off point beyond, i.e. after a block.



Moving a take-off point beyond a block

A block of $1/G$ has to be introduced in the feedback path to maintain the equivalence as shown in Fig.

Procedure for reduction of Block Diagram model

Step1: Reduce the cascade blocks.

Step2: Reduce the parallel blocks.

Step3: Reduce the internal feedback loops.

Step4: Shift take-off points towards right and summing points towards left.

Step 5: Repeat step 1 to step 4 until the simple form is obtained.

Step 6: Find transfer function of whole system as $\frac{C(s)}{R(s)}$

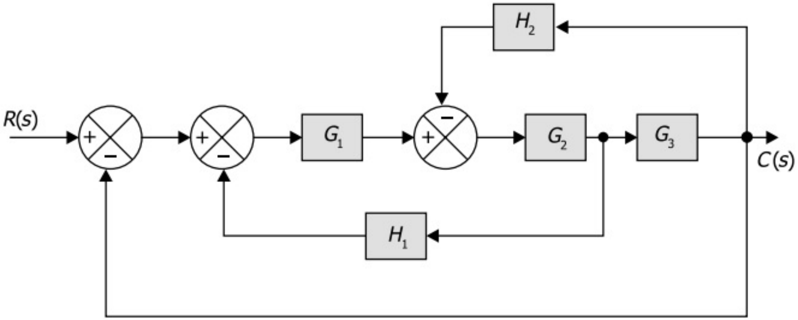
Procedure for finding output of Block Diagram model with multiple inputs

Step1: Consider one input taking rest of the inputs zero, find output using the procedure.

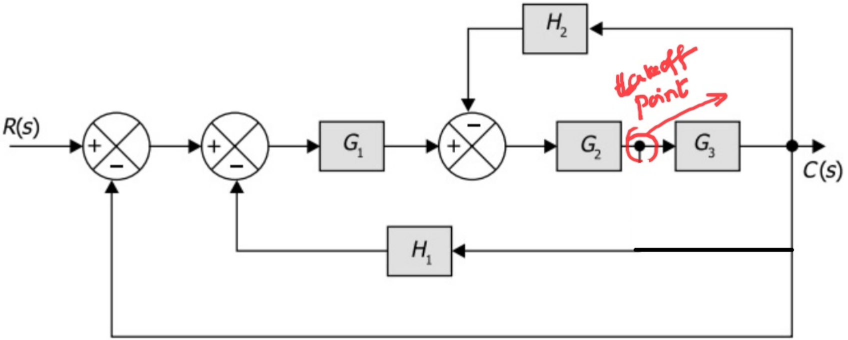
Step2: Follow step 1 for each inputs of the given Block Diagram model and find their corresponding outputs.

Step3: Find the resultant output by adding all individual outputs.

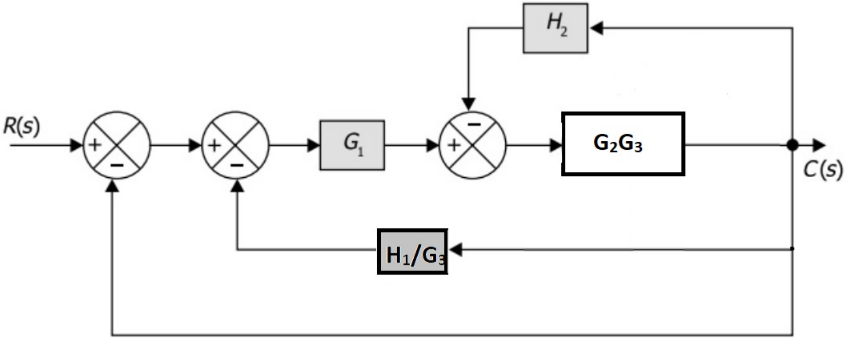
Problem1:Reduce the block diagram of Figure to canonical form and derive the overall transfer function



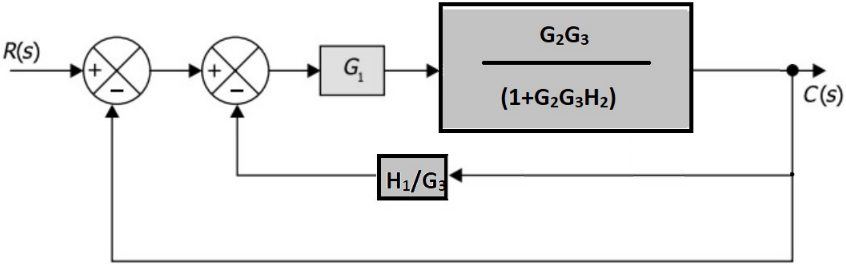
Problems on Block Reduction

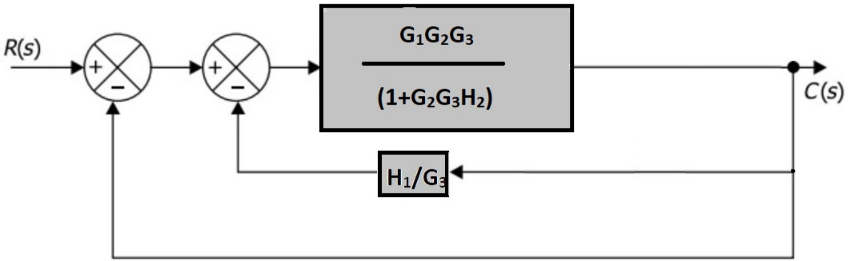


Problems on Block Reduction

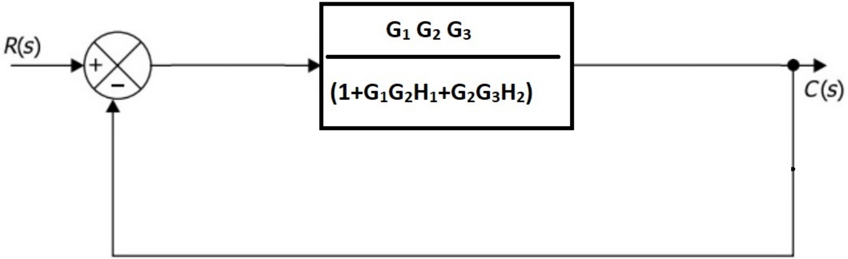


Problems on Block Reduction

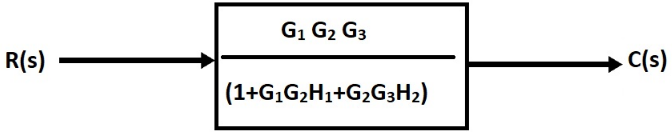



R(s)
+
−
+
−
G₁G₂G₃
(1+G₂G₃H₂)
C(s)
H₁/G₃

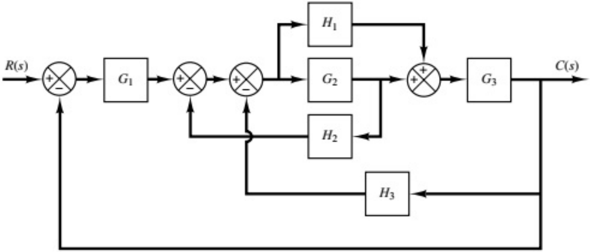
Problems on Block Reduction



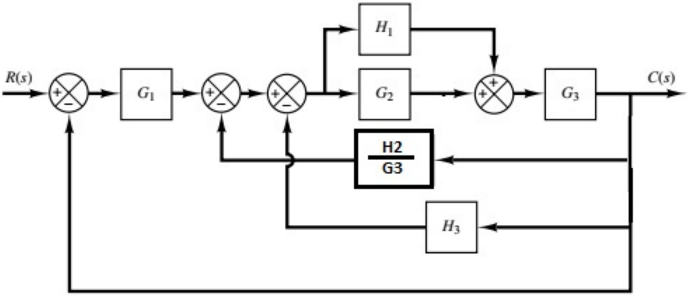
Problems on Block Reduction



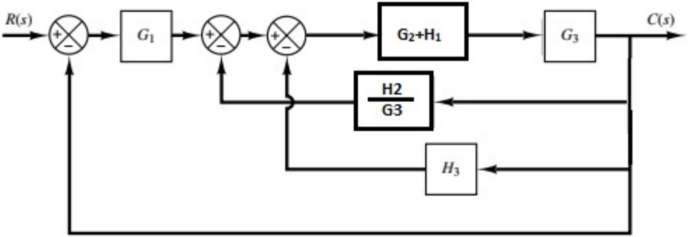
Problem2:Reduce the block diagram of Figure to canonical form and derive the overall transfer function

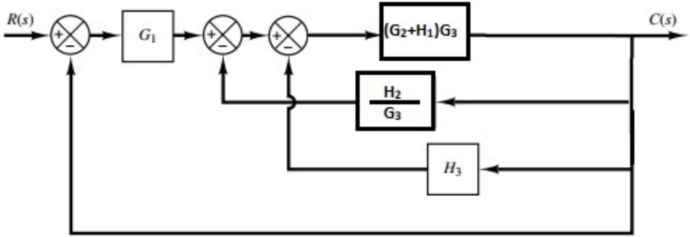


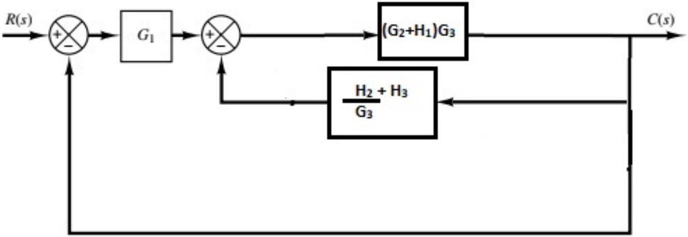
Problems on Block Reduction

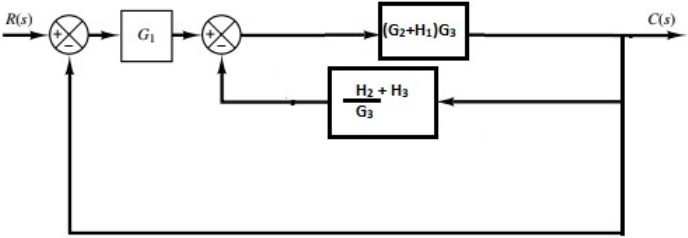


Problems on Block Reduction

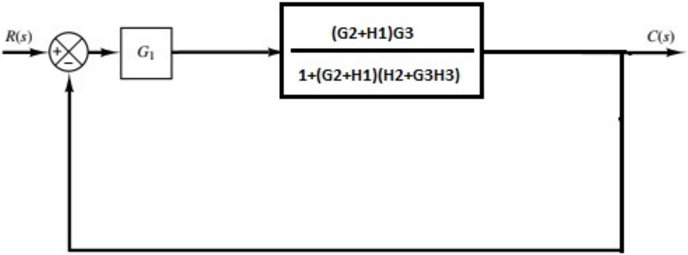



R(s)
G1
(G2+H1)G3
C(s)
H2
G3
H3

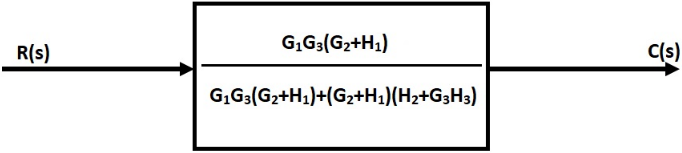

R(s)
+
−
G1
+
−
(G2+H1)G3
C(s)
H2 + H3
G3


R(s)
+
−
G1
+
−
(G2+H1)G3
C(s)
H2 + H3
G3

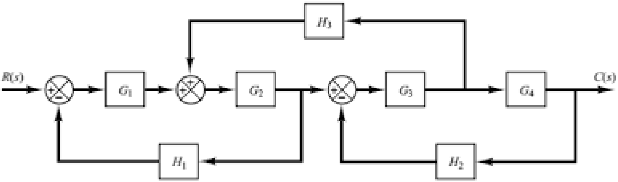
Problems on Block Reduction



Problems on Block Reduction



Problem3:Reduce the block diagram of Figure to canonical form and derive the overall transfer function



Problems on Block Reduction

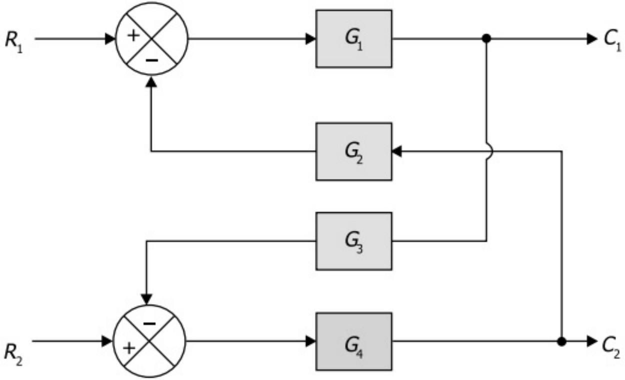
Problems on Block Reduction

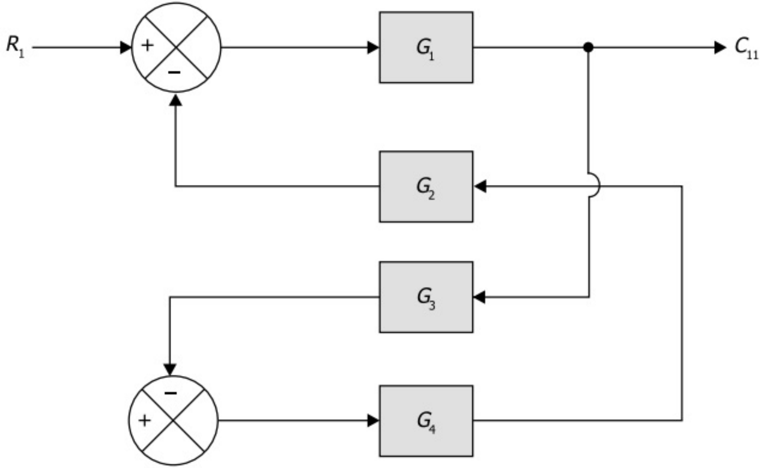
Problems on Block Reduction

Problems on Block Reduction

Problems on Block Reduction

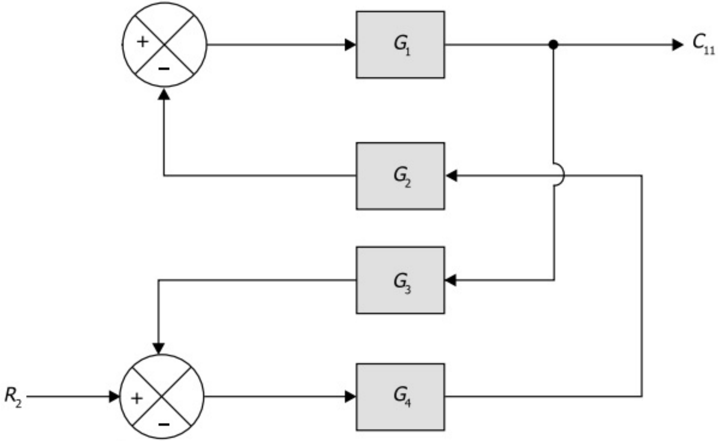
Reduce the block diagram of Figure to canonical form.




R_1
+
−
G_1
C_{11}
G_2
G_3
−
+
G_4

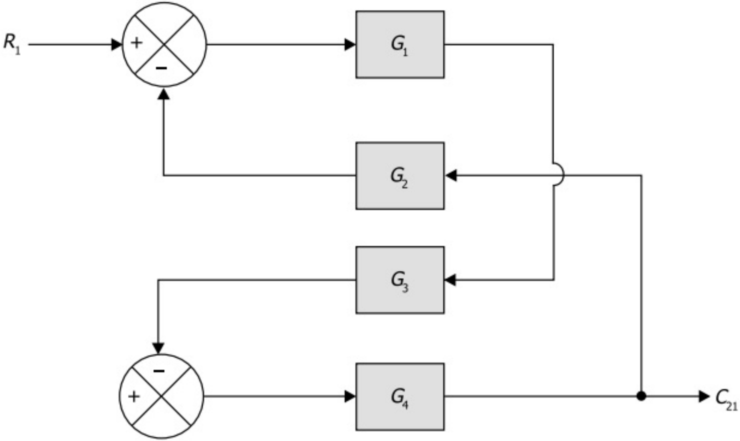
Problems on Block Reduction

Problems on Block Reduction

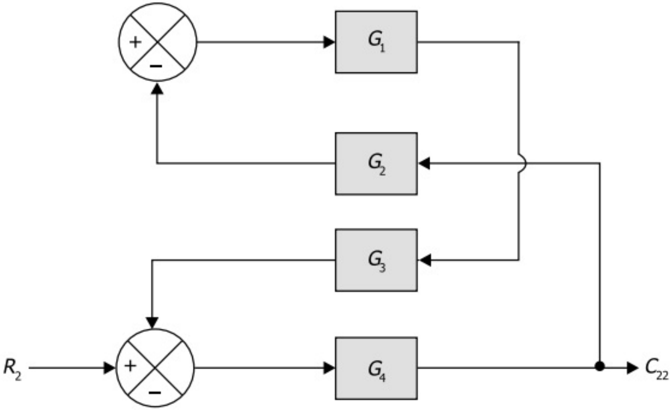


Problems on Block Reduction

Problems on Block Reduction

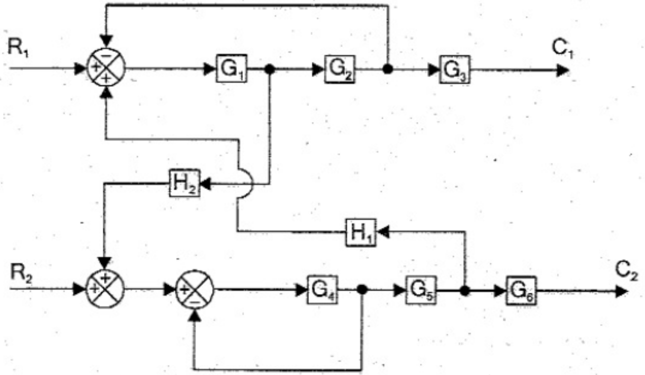


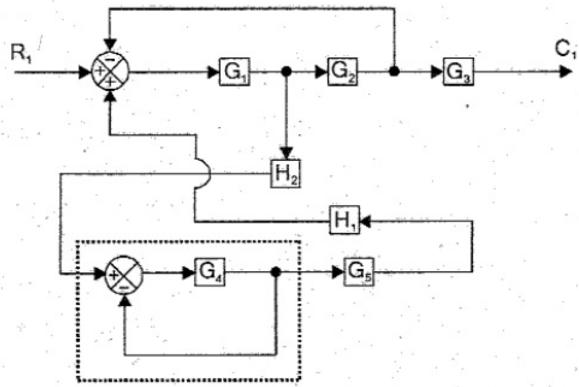
Problems on Block Reduction

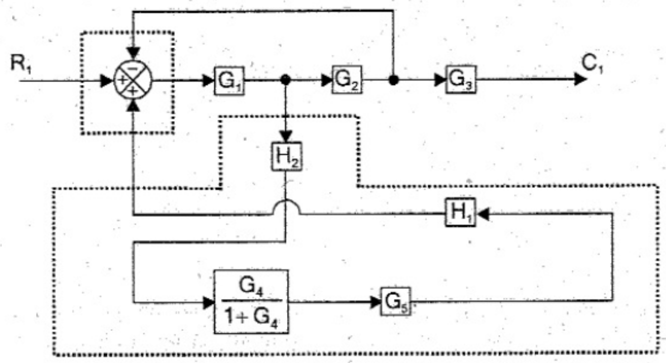


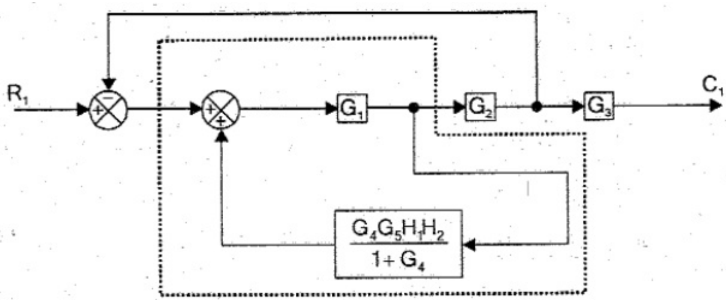
Problems on Block Reduction

For the system represented by block diagram shown in figure, determine C1/R1 and C2/R1.



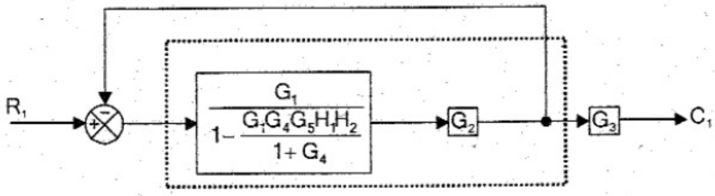

R1
C1
G1
G2
G3
H2
H1
G4
G5

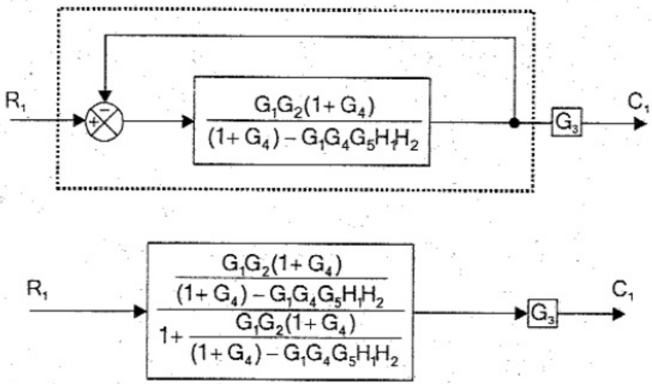

R1
C1
G1
G2
G3
H2
H1
G4
1+G4
G5
+
−
+

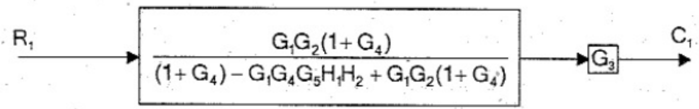


R1
C1
G1
G2
G3
$\frac{G_4 G_5 H_1 H_2}{1+G_4}$

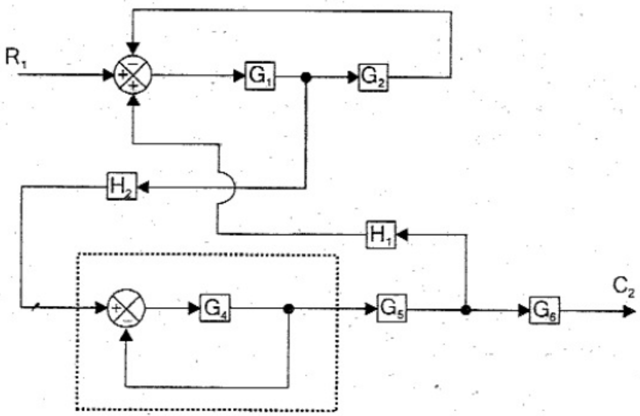
Problems on Block Reduction




R1
G1G2(1+G4)
(1+G4) − G1G4G5H1H2
G3
C1
R1
G1G2(1+G4)
(1+G4) − G1G4G5H1H2
1+ G1G2(1+G4)
(1+G4) − G1G4G5H1H2
G3
C1



$$\frac{C_1}{R_1} = \frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)(1+G_4) - G_1G_4G_5H_1H_2}$$


R1
+
−
+
G1
G2
H2
H1
+
G4
G5
G6
C2

Problems on Block Reduction

The transfer function of the system when the input and output are R_1 and C_1 is given by,

$$\frac{C_1}{R_1} = \frac{G_1G_2G_3(1+G_4)}{(1+G_1G_2)(1+G_4) - G_1G_4G_5H_1H_2}$$

The transfer function of the system when the input and output are R_1 and C_2 is given by,

$$\frac{C_2}{R_1} = \frac{G_1G_4G_5G_6H_2}{(1+G_4)(1+G_1G_2) - G_1G_4G_5H_1H_2}$$

Control Systems

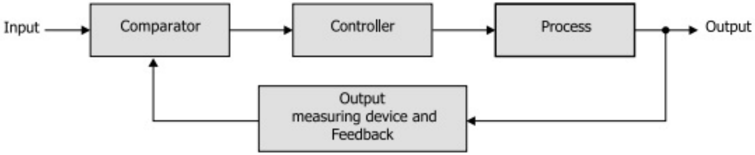
Unit I: EFFECT OF FEEDBACK ON SYSTEM PERFORMANCE

Sub-code: 18EE502
Department of Electrical Engineering
Bapatla Engineering College
Bapatla

September 28, 2021

FEEDBACK SYSTEM

A closed-loop or feedback system generates the output in response to an error signal obtained by comparing the input with the feedback signal produced by measuring output so that the error is continually reduced and the process comes under control.



FEEDBACK SYSTEM

Feedback control systems are used in all applications like in production control, quality control, economy control, process control, etc. Process control systems are feedback control systems where the output variables like temperature, pressure, humidity, etc. are regulated by feedback control mechanism.

EFFECT OF FEEDBACK

A feedback is provided to bring about improvement in the performance of a control system. The advantages of feedback in a control system are:

a) Feedback reduces the sensitivity of the system to its parameter variations. Parameters may vary due to ageing, environmental changes, etc. If feedback is introduced the system performance will not be adversely effected.

b) Feedback improves the sensitivity of a control system but there would be reduction in system gain.

c) Feedback improves the stability if properly designed.

d) Negative feedback reduces the overall gain of the system.

e) System response to disturbance signal can be reduced with feedback.

Effect of feedback on parameter Variations

Suppose due to system parameter changes, the transfer function G(s) changes to G(s) + ΔG(s), the corresponding change in output, i.e. ΔC(s) with respect to open-loop and closedloop system will be,

$$\Delta C(s) = \Delta G(s)\,R(s) \qquad \text{for open-loop system and}$$

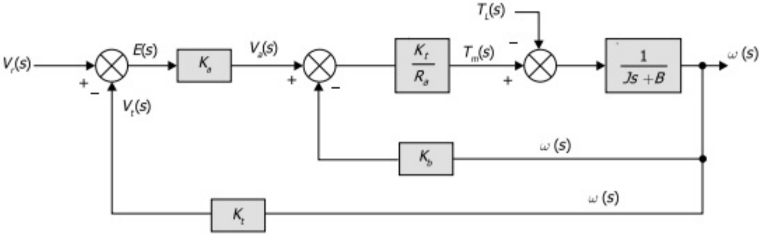
$$= \frac{\Delta G(s)R(s)}{1+G(s)H(s)} \qquad \text{for closed-loop system (ignoring change } \Delta G(s) \text{ in the denominator).}$$

So the change in output due to parameter variation of G(s) in the closed-loop (feedback) system is reduced by a factor of [1 + G(s) H(s)] which is usually much greater than unity

Sensitivity is the ratio of relative variation of the overall transfer function of the system due to variation of G(s). The effect of feedback in control system is to reduce the sensitivity to parameter variation on the system's output.

Effect of feedback on transient response

Transient response is the response of a system with respect to time before steady-state is reached. To understand the effect of feedback on transient response, we may again consider the speed control system



Applying a step input of $V_r(s) = \frac{K_2}{s}$, we get the speed of the motor for a closed-loop operation as

$$\omega(s) = \frac{KV_r(s)}{\tau s + KK_t + 1} = \frac{KK_2/\tau}{s\left(s + \frac{KK_t+1}{\tau}\right)}$$

Effect of feedback on transient response

Taking inverse Laplace transform, we get

$$\omega(t) = \frac{KK_2}{1+KK_t}\left[1 - e^{-\left(\frac{KK_t+1}{r}\right)t}\right] \quad \text{for a closed-loop system and}$$

$$= KK_2\left(1 - e^{-\frac{-t}{\tau}}\right) \quad \text{for an open-loop system,}$$

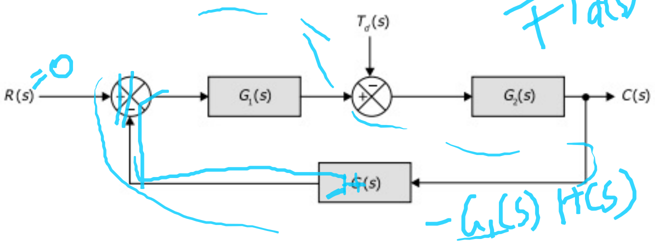
where and

$$\tau = \frac{R_a J}{R_a B + K_T K_b}$$

It is clear from the above expression of w(t) that the open-loop system with a large time constant t exhibits poor transient response. However in the closed-loop system, the time constant is reduced by a factor of (1 + KKt) and hence the transient response can be adjusted by varying amplifier gain Ka and tachometer gain Kt, if needed.

Effect of feedback on disturbance signal

A disturbance signal is an unwanted input signal that affects the system's output. The block diagram of a closed-loop system with disturbance signal Td(s) is shown in Fig.



The ratio of output to disturbance signal is obtained by putting $R(s) = 0$ in the above Fig.

$$\frac{C(s)}{T_d(s)} = \frac{-G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

Effect of feedback on disturbance signal

If $|G_1(s)G_2(s)H(s)| >> 1$ in the working range of frequencies, then

$$\frac{C(s)}{T_d(s)} = \frac{-G_2(s)}{1 + G_1(s)G_2(s)H(s)} = \frac{-1}{G_1(s)H(s)}$$

The effect of disturbance can be minimized through feedback if $G_1(s)$ is made sufficiently large. Again we may consider the example of the speed control system and comparing with above fig, we get

$$R(s) = V_r(s); C(s) = \omega(s)$$

$$G_1(s) = \frac{K_a K_T}{R_a}; G_2(s) = \frac{1}{J_s + B}$$

$$H(s) = K_t + \frac{K_b}{K_a}$$

Effect of feedback on disturbance signal

Thus the steady-state error for a closed-loop speed control system with $R(s) = V_r(s) = 0$ and $T_d(s) = A/s$ is given by

$$
\begin{aligned}
e_x^c &= \lim_{x \to 0} [R(s) - C(s)] = \lim_{x \to 0} s[-C(s)] \\
&= \lim_{x \to 0} \frac{sG_2(s)T_d(s)}{1 + G_1(s)G_2(s)H(s)} \\
&= \lim_{x \to 0} \frac{A}{1 + \frac{K_e K_T}{R_a}\left(\frac{1}{J_s + B}\right)\left(K_t + \frac{K_b}{K_a}\right)} \\
&= \frac{AR_a}{R_e B + K_r\left(K_b + K_a K_t\right)}
\end{aligned}
$$

Effect of feedback on steady-state error

From Fig. 5.3(a) we may write the error $E(s)$ for open and closed-loop system as follows.

$$E(s) = R(s) - C(s)$$
$$= [1 - G(s)]R(s) \quad \text{for open-loop system and}$$
$$= \frac{R(s)}{1 + G(s)} \quad \text{for closed-loop system with } H(s) = 1$$

So, the steady-state error for open-loop system with step input is

$$e_x^{at} = \lim_{n \to 0} SE(s)$$
$$= \lim_{s \to 0} s[1 - G(s)]\frac{1}{s}$$
$$= 1 - G(0)$$

Effect of feedback on steady-state error

The steady-state error for closed-loop system with step input is

$$\begin{aligned} e_s^c &= \lim_{x \to \infty} s[R(s) - C(s)] \\ &= \lim_{x \to \infty} \frac{1}{1 + G(s)} \left(\frac{1}{s} \right) \\ &= \frac{1}{1 + G(0)} \end{aligned}$$

$G(0)$ is often called the DC gain and is normally greater than unity. So, the steady-state error for an open-loop system will be of significant magnitude as compared to that for a closed-loop system with a reasonably large DC loop gain $G(0)$.

Effect of feedback on overall gain

$$\Delta C(s) = \Delta G(s)\, R(s) \qquad \text{for open-loop system and}$$

$$= \frac{\Delta G(s) R(s)}{1 + G(s)H(s)} \qquad \text{for closed-loop system (ignoring change } \Delta G(s) \text{ in the denominator).}$$

Thus, the gain of an open-loop system is G(s). When we use negative feedback, this gain gets reduced by a factor 1/[1 + G(s)H(s)]

Effect of feedback on stability

Use of feedback improves the stability of a system. From the transfer function, we can examine the location of the poles in the s-plane. If the poles get shifted more to the left-hand side of the imaginary axis, we can say that the system becomes more stable. For example, let us say that the span-loop transfer function $G(s)$ is $K/s + \tau$ The pole is located at $s = -\tau$. The overall transfer function of the system with unity negative feedback will be

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{K}{s} + \tau}{1 + \frac{K}{s+\tau}} = \frac{K}{s + (\tau + K)}$$

Now, the pole gets shifted to

$$s = -(T + K)$$

The pole gets shifted from $s = -\tau$ to $s = -(\tau + K)$. Thus, we can see that feedback can make the system more stable.

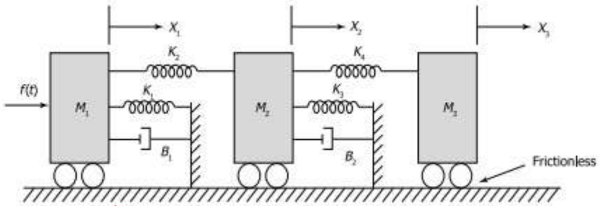
Control Systems

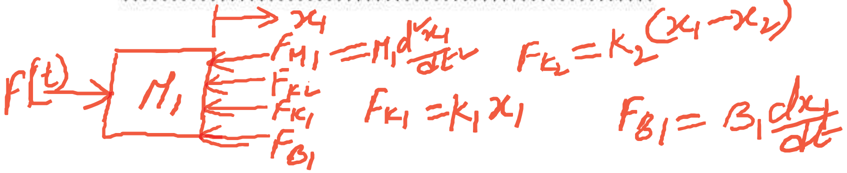
Unit I: Problems and Short answer Questions

Sub-code: 18EE502
Department of Electrical Engineering
Bapatla Engineering College
Bapatla

September 3, 2021

Prob2: Draw the mechanical network for the system shown in Fig and draw its analogous circuit.





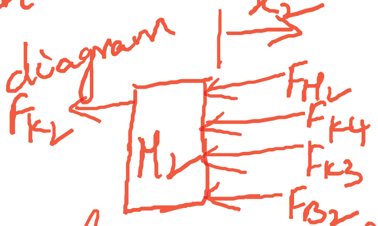
Problem2

Force balance eqn for M_1

$$F(t) = F_{M_1} + F_{K_2} + F_{K_1} + F_{B_1}$$

$$M_1 \frac{d^2x_1}{dt^2} + K_2(x_1 - x_2) + K_1 x_1 + B_1 \frac{dx_1}{dt} = f(t) \quad \text{—(1)}$$

free body diagram



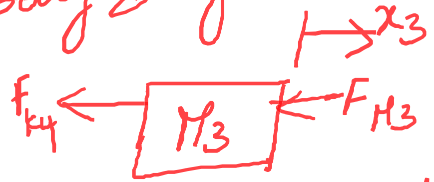
Force balance eqn for M_2 's

$$F_{K_2} + F_{M_2} + F_{K_4} + F_{K_3} + F_{B_2} = 0$$

Problem2

$$K_2(x_2-x_1)+M_2\frac{d^2x_2}{dt^2}+K_4(x_2-x_3)+K_3x_2+B_2\frac{dx_2}{dt}=0 \quad \text{—(2)}$$

free body Diagram

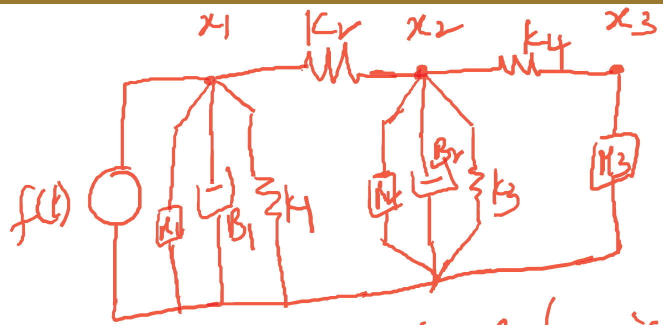


force balance eqn for M_3 is

$$F_{K4}+F_{M3}=0$$

$$K_4(x_3-x_2)+M_3\frac{d^2x_3}{dt^2}=0 \quad \text{—(3)}$$

Problem2



Equivalent ckt for Mechanical System

Force-current analogous

$M_1 \rightarrow C_1$; $B_1 \rightarrow \frac{1}{R_1}$; $K_1 \rightarrow \frac{1}{L_1}$; $K_2 \rightarrow \frac{1}{L_2}$

$M_2 \rightarrow C_2$; $B_2 \rightarrow \frac{1}{R_2}$; $K_3 \rightarrow \frac{1}{L_3}$

Problem2

$K_4 \Rightarrow \frac{1}{L_4}$; $M_3 \rightarrow C_3$

$$M_1 \frac{d^2 x_1}{dt^2} + K_2(x_1 - x_2) + K_1 x_1 + B_1 \frac{dx_1}{dt} = f(t) \quad \text{—(1)}$$

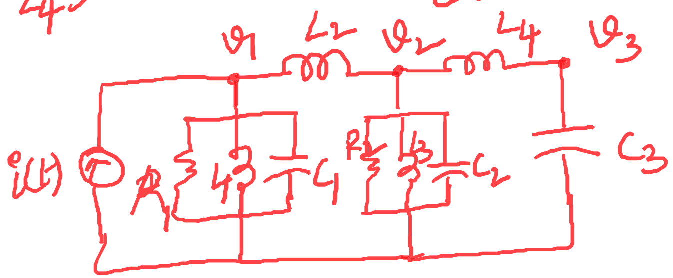
$$C_1 \frac{d^2 \varphi_1}{dt^2} + \frac{1}{L_2}(\varphi_1 - \varphi_2) + \frac{1}{L_1}\varphi_1 + \frac{1}{R_1}\frac{d\varphi_1}{dt} = i(t)$$

$$C_1 \frac{dv_1}{dt} + \frac{1}{L_2}\left(\int (v_1 - v_2)\,dt\right) + \frac{1}{L_1}\int v_1\,dt + \frac{1}{R} v_1 = i(t) \quad \text{—(4)}$$

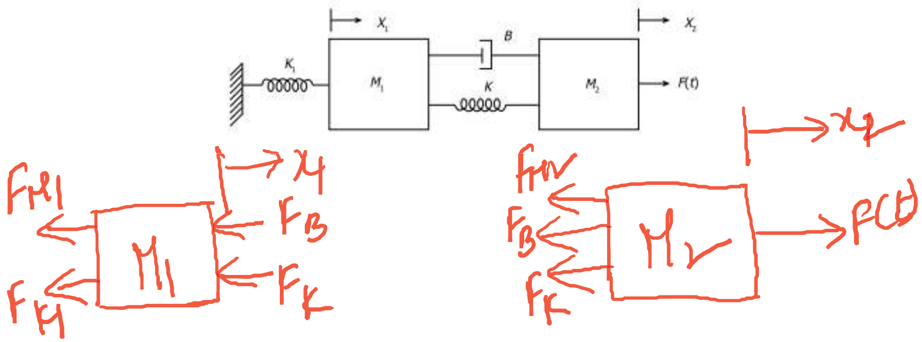
Problem2

$$\frac{1}{L_2}\int(v_2-v_1)dt + C_2\frac{dv_2}{dt} + \frac{1}{L_4}\int(v_2-v_3)dt + \frac{1}{L_3}\int v_2 dt + \frac{1}{R_2}v_2 = 0 \quad \text{---} (5)$$

$$\frac{1}{L_4}\int(v_3-v_2)dt + C_3\frac{dv_3}{dt} = 0 \quad \text{---} (6)$$



Prob4: Using force–voltage analogy and force–current analogy draw the equivalent analogous system for the system shown in Figure.



Problem4

force balance eqn for M_1

$$F_{M_1} + F_{K_4} + F_B + F_K = 0 \quad \text{——(1)}$$

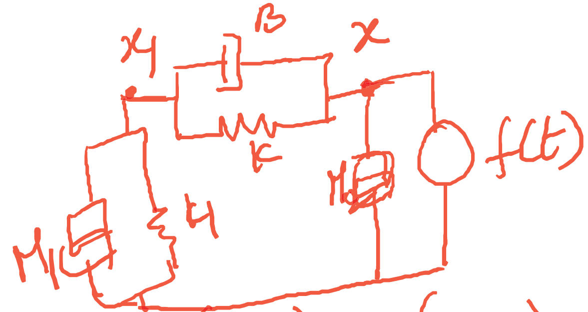
$$M_1 \frac{d^2 x_4}{dt^2} + K_4 x_4 + B\frac{d(x_4 - x)}{dt} + K(x_4 - x) = 0 \quad \text{——(2)}$$

force balance eqn for M_2

$$F_{M_2} + F_B + F_K = f(t) \quad \text{——(3)}$$

$$M_2 \frac{d^2 x}{dt^2} + B\frac{d(x - x_4)}{dt} + K(x - x_4) = f(t) \quad \text{——(4)}$$

Problem4



$$M_1 \frac{d^2 x_1}{dt^2} + K_1 x_1 + B \frac{d(x_1 - x)}{dt} + K(x_1 - x) = 0 \quad \text{—(2)}$$

F-V

$$L_1 \frac{d^2 q_1}{dt^2} + \frac{1}{C_1} q_1 + R \frac{d(q_1 - q)}{dt} + \frac{1}{C}(q_1 - q) = 0 \quad \text{—(3)}$$

Problem4

$$L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1\,dt + R(i_1 - i) + \frac{1}{C}\int (i_1 - i)\,dt = 0 \quad \text{— (6)}$$

$$M_2 \frac{d^2x}{dt^2} + B\frac{d(x-x_1)}{dt} + K(x-x_1) = f(t) \quad \text{— (4)}$$

$$L_2 \frac{d^2q}{dt^2} + R\frac{d(q-q_1)}{dt} + \frac{1}{C}(q-q_1) = e(t) \quad \text{— (7)}$$

$$L_2 \frac{di}{dt} + R(i-i_1) + \frac{1}{C}\int (i-i_1)\,dt = e(t) \quad \text{— (8)}$$

Problem4

F-V equivalent ckt

Take eqt (6) & (8), we draw

Analogous ckt (F-V)

Problem4

F - (I)

$v_c = \frac{1}{C}\int i\,dt \Rightarrow i_c = C\frac{dv}{dt}$

$M_1 \frac{d^2x_1}{dt^2} + K_1 x_1 + B\frac{d(x_1 - x)}{dt} + K(x_1 - x) = 0$ —(8)

$C_1 \frac{d^2\psi_1}{dt^2} + \frac{1}{L_1}\psi_1 + \frac{1}{R}\frac{d(\psi_1 - \psi)}{dt} + \frac{1}{L}(\psi_1 - \psi) = 0$ —(9)

$C_1 \frac{dv_1}{dt} + \frac{1}{L_1}\int v_1\,dt + \frac{1}{R}(v_1 - v) + \frac{1}{L}\int (v_1 - v)\,dt = 0$ —(10)

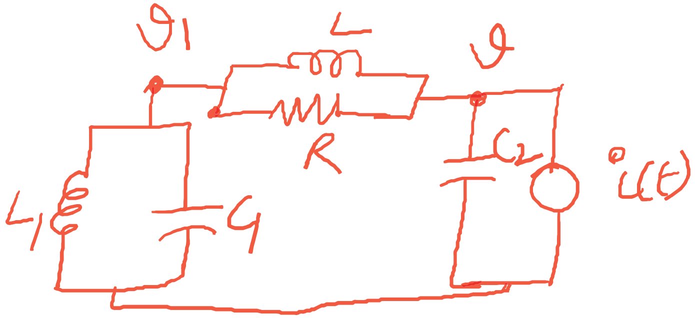
Problem4

$$M_2\frac{d^2x}{dt^2}+B\frac{d(x-x_1)}{dt}+K(x-x_1)=f(t) \quad (4)$$

$$C_2\frac{d^2\psi}{dt^2}+\frac{1}{R}\left(\frac{d(\psi-\psi_1)}{dt}\right)+\frac{1}{L}(\psi-\psi_1)=i(t) \quad (1)$$

$$C_2\frac{dv}{dt}+\frac{1}{R}(v-v_1)+\frac{1}{L}\int(v-v_1)dt=i(t) \quad (2)$$

Problem4



F-I analogous circuit

Control Systems

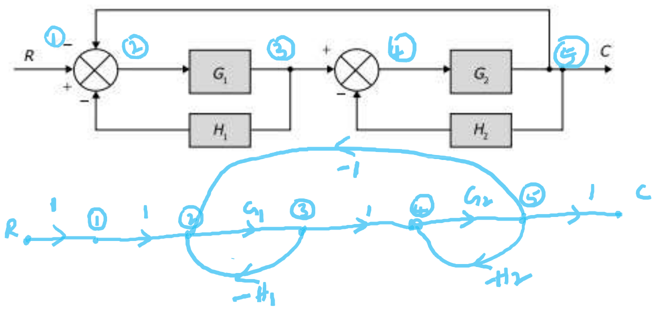
Unit I: Problems on Signal Flow Graph

Sub-code: 18EE502
Department of Electrical Engineering
Bapatla Engineering College
Bapatla

September 23, 2021

Example

Obtain the transfer function of the system shown in Figure by block diagram reduction technique and also using signal flow graph.



Problem

Step 1:

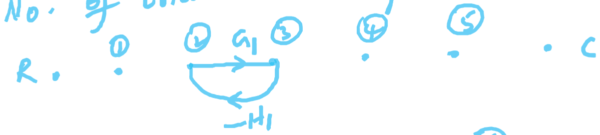
No of forward paths = 1

forward path gain $= 1 \times 1 \times G_1 \times 1 \times G_2 \times 1$

$= G_1 G_2$

Step 2:

No. of individual loops = 3



Problem

$P_{11} = -G_1H_1$

$P_{12} = -G_2H_2$

$P_{13} = G_1 \times 1 \times G_2 \times -1 = -G_1G_2$

Step3 :

No. of 2 non-touching loops = 1

$$P_{21} = G_1 \times -H_1 \times G_2 \times -H_2 = G_1G_2H_1H_2$$

Problem

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

where $\Delta = 1 - \sum$ single nontouching loop gains
$+ \sum$ 2 nontouching loop gains
$- \sum$ 3 nontouching loop gains
- - -

P_K = K^{th} forward path gain.

$$T = \frac{P_1}{\Delta} = \frac{P_1}{1 - \sum_{m=1}^{3} P_{1m} + \sum_{n=1} P_{2n}}$$

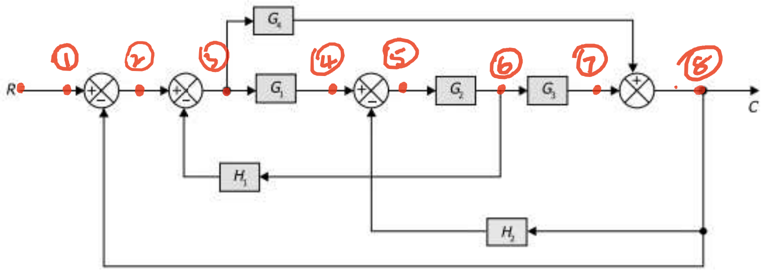
Problem

$$T = \frac{G_1G_2}{1-(-G_1H_1-G_2H_2-G_1G_2)+G_1G_2H_1H_2}$$

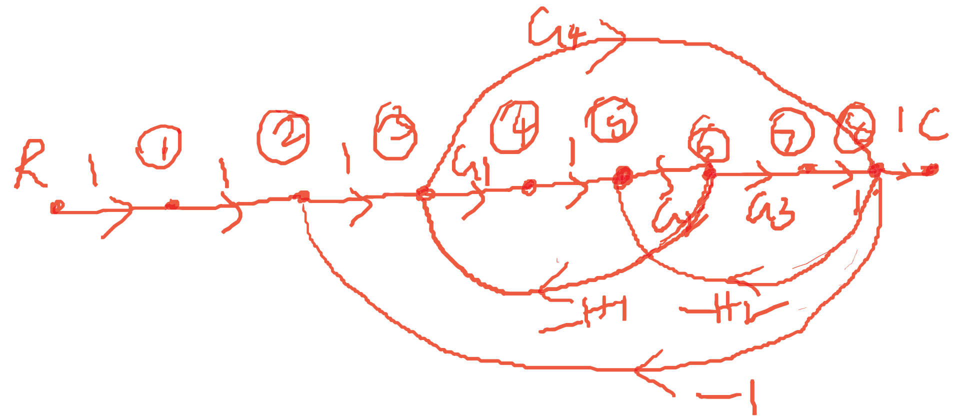
$$T = \frac{G_1G_2}{1+G_1H_1+G_2H_2+G_1G_2+G_1G_2H_1H_2}$$

Problem1

Obtain the transfer function of the system shown in Figure using signal flow graph.



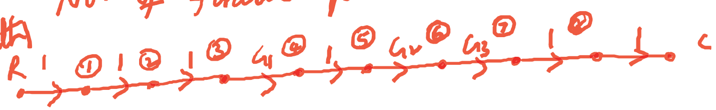
Problem



step1:

No. of forward paths = 2

Path 1



Problem

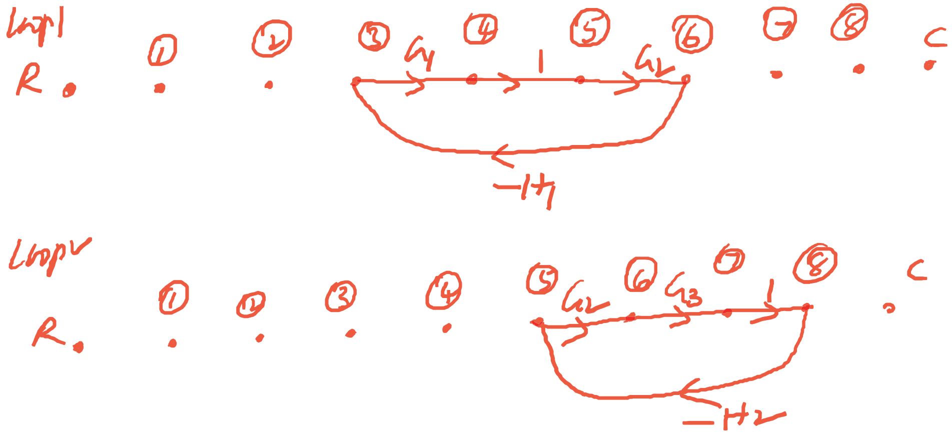
Path 2

$P_1 = G_1 G_2 G_3$

$P_2 = G_4$

Step 2: No. of individual loops = 3

Problem



Problem

Loops

Step3: No. of 2 non touching loops = 0

$P_{11} = -G_1G_2H_1$

$P_{12} = -G_2G_3H_2$

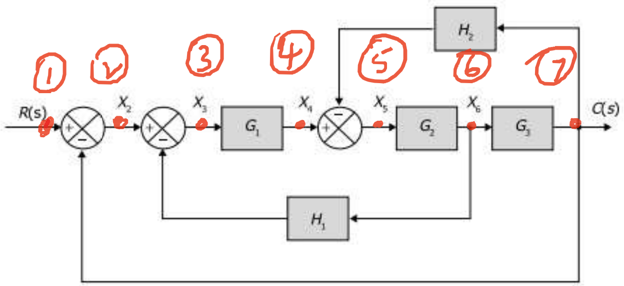
$P_{13} = -G_1G_2G_3$

$$T = \frac{P_1 + P_2}{1 - P_{11} - P_{12} - P_{13}}$$

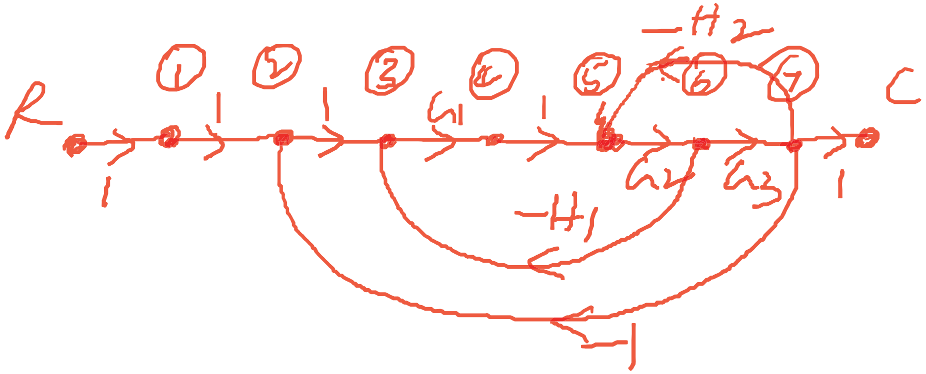
$$T = \frac{G_1G_2G_3 + G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$$

Problem2

Find the gain of the control system represented in block diagram form as in Figure using Mason's gain formula.



Problem



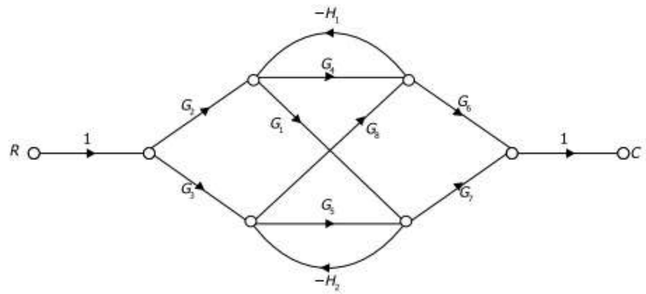
Problem

Problem

Problem

Example

Find the transfer function of the system shown in Figure



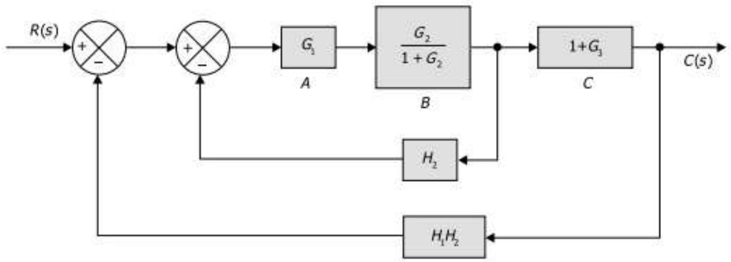
Problem

Problem

Problem

Problem2

Use Mason's gain formula for determining the overall transfer function of the system shown in Figure.



Control Systems

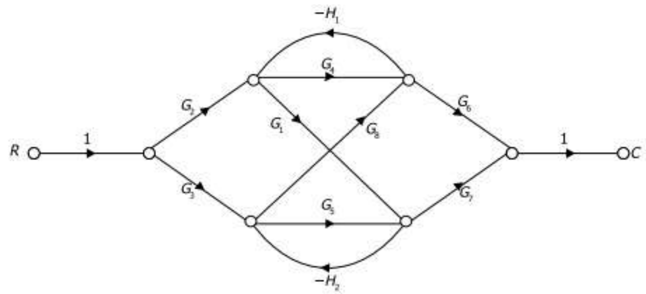
Unit I: Problems on Signal Flow Graph

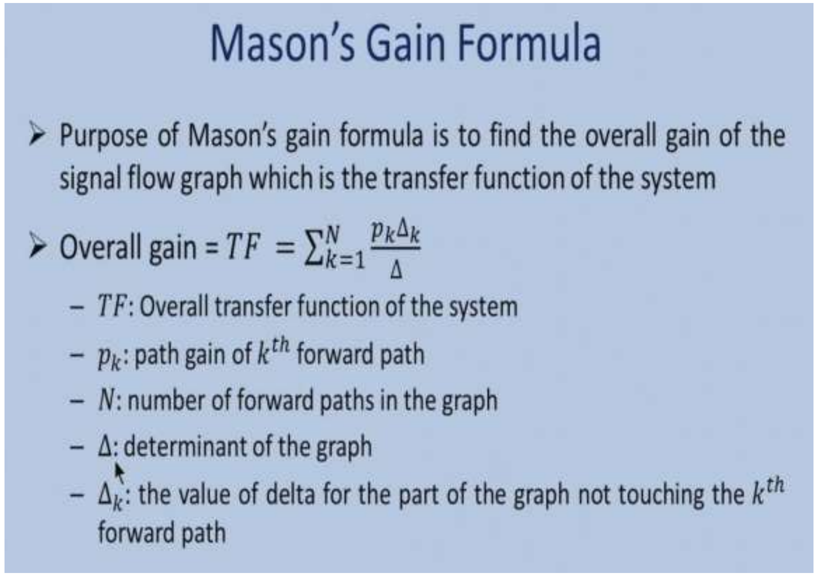
Sub-code: 18EE502
Department of Electrical Engineering
Bapatla Engineering College
Bapatla

September 24, 2021

Example

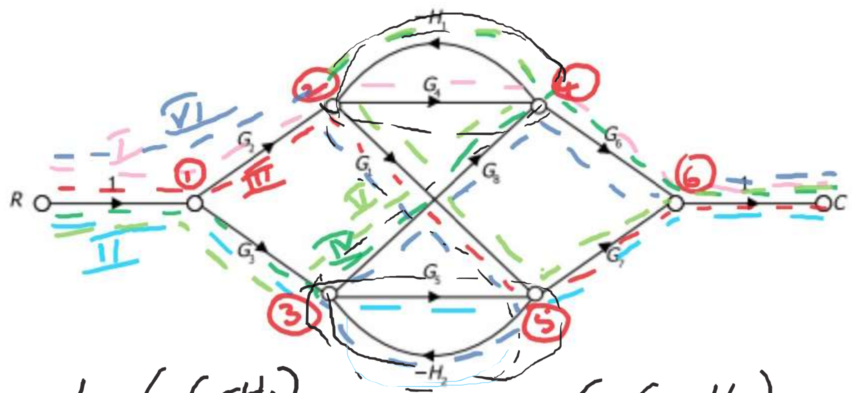
Find the transfer function of the system shown in Figure




Mason's Gain Formula
Purpose of Mason's gain formula is to find the overall gain of the signal flow graph which is the transfer function of the system
Overall gain = $TF = \sum_{k=1}^{N} \frac{p_k \Delta_k}{\Delta}$
– TF: Overall transfer function of the system
– p_k: path gain of k^{th} forward path
– N: number of forward paths in the graph
– Δ: determinant of the graph
– Δ_k: the value of delta for the part of the graph not touching the k^{th} forward path

Problem

(1) 0 3 closed loops



$\Delta_1 = 1 - (-G_5 H_2)$; $\Delta_2 = 1 - (-G_4 H_1)$

$\Delta_3 = 1$; $\Delta_4 = 1$; $\Delta_5 = 1$; $\Delta_6 = 1 - 0 = 1$

Problem

step1 :

No. of forward paths = 6

1st path gain $P_1 = G_2 G_4 G_6$
Δ_1

2nd forward path gain $P_2 = G_3 G_5 G_7$

3rd forward path gain $P_3 = G_2 G_1 G_7$

Problem

4th forward path gain (P_4) = $G_3 G_8 G_6$

5th forward path gain (P_5) = $-G_3 G_8 H_1 G_1 G_7$

6th forward path gain (P_6) = $-G_2 G_1 H_2 G_8 G_6$

step 2: No. of individual loops = 3

1st loop gain = $-G_4 H_1$; 2nd loop gain = $-G_5 H_2$

3rd loop gain = $G_1 G_8 H_1 H_2$

step 3: No. of two nontouching loops = 1

Two nontouching loop gain = $(-G_4 H_1)(-G_5 H_2)$ = $G_4 G_5 H_1 H_2$

$$T = \frac{1}{\Delta} \sum_{K=1}^{6} P_K \Delta_K$$

Problem

$$T = G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_8G_6 - G_3G_8H_1G_1G_7 - G_3G_1G_8G_6H_2$$

$$T = \frac{G_2G_4G_6(1+G_5H_2) + G_3G_5G_7(1+G_4H_1) + G_1G_2G_7 + G_3G_8G_6 - G_3G_8H_1G_1G_7 - G_3G_1G_8G_6H_2}{1+G_4H_1+G_5H_2+G_1G_8H_1H_2 + G_4G_5H_1H_2}$$

Control Systems

Unit II: Time Domain Analysis

Sub-code: 18EE502
Department of Electrical Engineering
Bapatla Engineering College
Bapatla

October 9, 2021

Introduction

Time response analysis is also called time domain analysis. Here, we study the response, i.e. the output as a function of time. Total time response c(t) of a control system consists of transient response ct(t) and steady state response css (t).

$c(t) = ct(t) + css(t)$

where $c(t)$ = total time response

$ct(t)$ = transient response

$css(t)$ = steady-state response.

The transient state of the system remains for a very short time while steady-state is that stage of the system as time t approaches infinity. A feedback control system has the inherent capabilities that its parameters can be adjusted to alter both its transient and steady-state behaviour.

Introduction

In order to analyse the transient and steady-state behaviour of control systems, we obtain a mathematical model of the system. For any specific input signal, a complete time response expression can then be obtained through the Laplace transform inversion. This expression yields the steady-state behaviour of the system with time tending to infinity. In case of simple deterministic signals, steady-state response expression can be calculated by the use of the final value theorem.

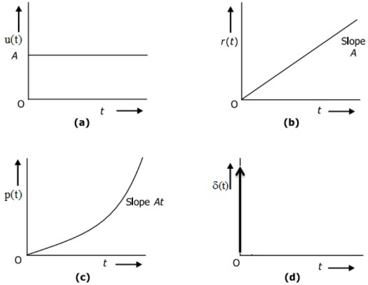
Introduction

In analysing or designing a particular control system, we must have a basis of comparison of performance of various control systems. This basis may be set by specifying particular test input signals and by comparing the responses of various systems to these input signals. Usually the input signals to control systems are not fully known.

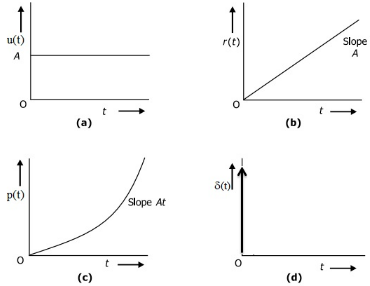
From experience it has been observed that the actual input signals which severely strain a control system are: a sudden shock, a sudden change, a constantly increasing change or a constantly accelerating change. Therefore, system dynamic behaviour for analysis and design can be studied and compared under application of standard test signals such as an impulse signal (sudden shock), a step signal (sudden change), a ramp signal (constant velocity) or a parabolic signal (constant acceleration). Sinusoidal signal is also another important test signal. With these test signals, mathematical and experimental analysis of control systems can be carried out easily since these signals are very simple functions of time.

Standard test signals

1. Step Signal: A step signal gives an instantaneous change in the value of the reference u(t) as shown in Fig.
2. Ramp Signal: A ramp signal gives a constant change in the value of the reference variable r(t) with respect to time, as shown in Fig. It is also the integral of a step signal.



3. Parabolic Signal: A parabolic signal gives an accelerating change in the value of the reference variable p(t). This is the integral of ramp signal. It is shown in Fig.
4. Impulse Signal: The unit-impulse signal gives an infinite magnitude to the value of the reference variable at t = 0 and a zero value everywhere except at t = 0.



Control systems may be of different orders viz. zero order system, first order system, second order system, and so on. The generalized relation between a particular input, say qi and the corresponding output, say q0, with proper simplified assumptions, can be written as

$$a_n \frac{d^n q_0}{dt^n} + a_{n-1} \frac{d^{n-1} q_0}{dt^{n-1}} + \ldots + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + \ldots + b_1 \frac{d}{d}$$

where q_0 is the output quantity q_i is the input quantity t is the time a 's and b 's are combination of system parameters which are assumed to be constants.

Taking Laplace transform and assuming all initial conditions to be zero, we get

$$\left[a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right] Q_0(s) = \left[b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right] Q_i(s)$$

Transfer function, $G(s) = \frac{Q_0(s)}{Q_i(s)} = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right)}$

The T.F. can also be written as,

$$= \frac{k\left(\tau_a s + 1\right)\left(\tau_b s + 1\right)\ldots}{s^N\left(\tau_1 s + 1\right)\left(\tau_2 s + 1\right)\ldots}$$

The power of s in the denominator determines the order of the system.

Order of a system

In the generalized system equation, i.e. equation if all the a's and b's are made zero except a0 and b0, we will get Zero order system. Therefore, q0 = kqi represents a zero order system. A simple example of a zero order system is a potentiometer where the output voltage is a fraction of the input voltage, i.e. e0 = kei.

If in the generalized system equation, if all a's and b's other than a1, a0 and b0 are taken as zero, we will get First order system.

$$\tau \frac{dq_0}{dt} + q_0 = kq_i$$

where,

$$\tau = \frac{a_1}{a_0} \text{ is called the time constant}$$

$$k = \frac{b_0}{a_0} \text{ is called the static sensitivity}$$

Order of a system

Any system that follows the above relation of equation is called a first order system. Taking Laplace transform,

$$\tau s Q_0(s) + Q_0(s) = kQ_i(s)$$

$$\text{Transfer Function,} \quad G(s) = \frac{Q_0(s)}{Q_i(s)} = \frac{k}{\tau s + 1}$$

A simple $R - C$ network having an input e_i and output across C as e_0 will have a similar transfer function and therefore can be called a first order system.

Order of a system

In the generalized equation if all a's and b's are made zero except a2, a1, a0 and b0 then we will get the equation for the second order system. Accordingly a second order system is one which follows the equation,

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

$$\frac{a_2}{a_0} \frac{d^2 q_0}{dt^2} + \frac{a_1}{a_0} \frac{dq_0}{dt} + q_0 = \frac{b_0}{a_0} q_i$$

$$\frac{1}{\omega_n^2} \frac{d^2 q_0}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dq_0}{dt} + q_0 = k q_i$$

where

$$\omega_n = \sqrt{\frac{a_0}{a_2}}, \text{ is called the undamped natural frequency}$$

$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$, is called the damping ratio.

$$\left[\because \frac{a_1}{a_0} = \frac{2\zeta}{\omega_n}, \therefore \zeta = \frac{a_1\omega_n}{2a_0} = \frac{a_1\sqrt{a_0}}{2a_0\sqrt{a_2}} = \frac{a_1}{2\sqrt{a_0 a_2}}\right]$$

Taking Laplace transform of equation (6.4), we get

$$\left[\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1\right] Q_0(s) = KQ_i(s)$$

$$\text{T.F. } G(s) = \frac{Q_0(s)}{Q_i(s)} = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1}$$

$$= \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

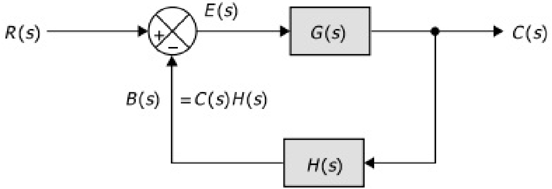
A mass spring damper system is an example of second order system.

Steady-State Error

The difference between the desired response and the actual response of a system is called the error. The error, if any, when the system settles down or stabilizes is called the steady-state error.
Transient state refers to the oscillatory condition of the system output, i.e. during the transient time before the system comes to final steady-state condition. During design stage, a system is tested for its steady-state and transient state errors in simulated condition. Modifications are made in the system parameters including the amplifier gain setting so as to obtain the desired steady state and transient state performance of the system. The error is measured in a simulated condition by applying certain test signals described earlier.

Steady-State Error

Steady-state error is a measure of the accuracy of a control system. The steady-state error of a control system is judged by the steady-state error due to step, ramp or acceleration input. Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input but the same system may exhibit non-zero, i.e. some steady-state error to a ramp input. Generally, steady state error should be as low as possible.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$E(s) = R(s) - B(s)$$

$$E(s) = R(s) - C(s)H(s).$$

Dividing both sides by $R(s)$

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)H(s)}{R(s)} = 1 - \frac{C(s)}{R(s)}H(s) = 1 - \frac{G(s)H(s)}{1 + G(s)H(s)}$$

or,

$$\frac{E(S)}{R(s)} = \frac{1}{1 + G(s)H(s)}$$

Steady-State Error

$$\therefore \text{ Error, } E(S) = \frac{R(s)}{1+G(s)H(s)}$$

The steady-state error is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ [from Final Value Theorem]

$$e_{ss} = \lim_{t\to\infty} e(t)$$

So, steady state error

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \lim_{s\to o} sE(s) \\ &= \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)} \end{aligned}$$

Static Position Error

The steady-state error of the system for a unit step input is

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s} \quad \left[R(s) = \frac{1}{s} \text{ for unit step input} \right]$$

$$e_{ss} = \lim_{s\to 0} \frac{1}{1+G(s)H(s)} = \frac{1}{1+K_p}$$

where, $K_p = \lim_{s\to 0} G(s)H(s)$; K_p is called the static position error coefficient. Now, we shall find the value of K_p for different types of systems, that is, type 0, type 1, type 2. Type is defined as the number of open-loop poles at the origin and is indicated by power of s, i.e. s^N in the denominator of the transfer function. For type 0 system, $s^0 = 1$; for type 1 , $s^1 = s$; for type 2, $s^2 = s^2$, and so on.

i) For a type 0 system,

$$G(s)H(s) = \frac{K\left(T_a s+1\right)\left(T_b s+1\right)\ldots}{s^0\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots} \text{ (from general equation)}$$

Static Position Error

$$K_p = \lim_{s\to 0} G(s)H(s) = \frac{K(0+1)(0+1)\ldots}{(0+1)(0+1)\ldots}$$

$$= \frac{K}{s^0} = K$$

ii) For type 1 or higher system,

$$G(s)H(s) = \frac{K\left(T_a s+1\right)\left(T_b s+1\right)\ldots}{s^N\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots}$$

$$K_p = \lim_{s\to 0} G(s)H(s) = \frac{K(0+1)(0+1)\ldots}{0(0+1)(0+1)\ldots.}$$

$$K_p = \infty \quad (\text{ for } N \geq 1)$$

Static Position Error

Now, we shall find the static position error for different systems at steady state.
i) For type 0 system,

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K} \quad [\because K_p = K]$$

ii) For type 1 or higher systems,

$$\begin{aligned} e_{ss} &= \frac{1}{1+K_p} \\ &= \frac{1}{1+\infty} = 0 \end{aligned}$$

Thus, for a unit step input, steady-state error for different types of systems is finite. Hence, every type of system is capable of following step input. Though type 0 system shows some error, higher type of systems can respond to step input very accurately.

Static velocity error

Static velocity error coefficient is associated with e_{ss} for unit ramp input. The steady-state actuating error of the system with a unit ramp input (unit velocity input) is given by

$$
\begin{aligned}
e_{ss} &= \lim_{s\to 0} \frac{\mathrm{sR}(s)}{1+G(s)H(s)} \\
&= \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s^2} \qquad \left[\because R(s) = \frac{1}{s^2}\right] \\
&= \lim_{s\to 0} \frac{1}{s+sG(s)H(s)} = \frac{1}{K_v}
\end{aligned}
$$

where, $K_v = \lim_{s\to 0} sG(s)H(s); K_v$ is called the static velocity error coefficient. Now, we shall find the static velocity error coefficient for different types of systems.

Static velocity error

i) For a type 0 system,

$$G(s)H(s) = \frac{K\left(T_a s+1\right)\left(T_b s+1\right)\ldots}{s^0\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots}$$

$$\therefore \quad = \lim_{s\to 0} sG(s)H(s)$$

$$= \lim_{s\to 0} \frac{sK\left(T_a s+1\right)\left(T_b s+1\right)\ldots\cdot}{\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots} = 0$$

ii) For a type 1 system,

$$K_v = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} s\frac{K\left(T_a s+1\right)\left(T_b s+1\right)\ldots}{s\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots} = K$$

ii) For a type 2 or higher system,

$$K_v = \lim_{s\to 0} SG(s)H(s) = \lim_{s\to 0} s\frac{K\left(T_a s+1\right)\left(T_b s+1\right)\ldots}{s^{N-1}\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots}$$

$$= \lim_{s\to 0} s\frac{K\left(T_a s+1\right)\left(T_b s+1\right)\ldots}{s^{N-1}\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots} = \infty$$

Static velocity error

Now, we shall find static velocity error for different types of systems at steady state. i) For type 0 system,

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty \quad [\because K_v = 0]$$

ii) For type 1 system,

$$e_{ss} = \frac{1}{K_r} = \frac{1}{K} \quad [\because K_v = K]$$

ii) For a type 2 or higher systems,

$$e_{ss} = \frac{1}{K_v} = \frac{1}{0} = \infty = 0 \quad [\because K_v = \infty]$$

For ramp input, steady-state error for type 0 system is infinite. Hence, a type 0 system is hot capable of following ramp input. The static velocity error for type 1 system is finite. But static velocity error for type 2 system or higher system is zero. So type 2 or higher systems are capable of following a ramp input very accurately. As we move from type 0 to type 2 or higher systems, the static velocity error goes on decreasing.

Static acceleration error

Static acceleration error coefficient is associated with e_{ss} for unit parabolic input. The steady state actuating error of the system with unit-parabolic input (acceleration) is given by

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$
$$= \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s^3} \qquad \left[\because R(s) = \frac{1}{s^3}\right]$$
$$= \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)H(s)} = \frac{1}{K_e}$$

where $\quad K_a = \lim_{s\to\infty} s^2 G(s)H(s)$

Now, we shall find the static acceleration error coefficient for different types of systems.

i) For type 0 system,

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2 K\left(T_a s+1\right)\left(T_b s+1\right)\ldots}{s^0\left(T_1 s+1\right)\left(T_2 s+1\right)\ldots} = 0$$

Static acceleration error

ii) For a type 1 system,

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) = \lim_{n \to 0} \frac{s^2 K (T_a s + 1)(T_b s + 1)}{s (T_1 s + 1)(T_2 s + 1) \ldots}$$

$$= \lim_{s \to 0} \frac{sK (T_a s + 1)(T_b s + 1) \ldots}{(T_1 s + 1)(T_2 s + 1) \ldots} = 0$$

iii) For a type 2 system

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) = \lim_{s \to 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \ldots}{s^2 (T_1 s + 1)(T_2 s + 1) \ldots} = K$$

iv) For a type 3 or higher system,

$$K_a = \lim_{s \to 0} s^2 G(s)H(s) = \lim_{n \to 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \ldots}{s^N (T_1 s + 1)(T_2 s + 1) \ldots}$$

$$= \lim_{s \to 0} \frac{K (T_a s + 1)(T_b s + 1) \ldots}{s^{N-2} (T_1 s + 1)(T_2 s + 1)} = \infty \quad [\because N \geq 3]$$

Static acceleration error

Now we shall find the static acceleration error for different types of systems at steady state.

i) For a type 0 system, $e_{ss} = \frac{1}{K_a} = \infty \quad [\therefore K_a = 0]$

ii) For a type 1 system, $e_{ss} = \frac{1}{K_a} = \infty \quad [\therefore K_a = 0]$

iii) For a type 2 system, $e_{ss} = \frac{1}{K_a} = \frac{1}{K} \quad [\therefore K_a = K]$

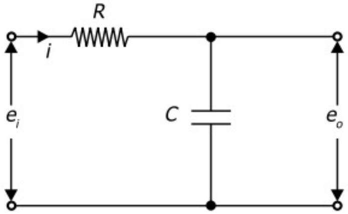
iv) For a type 3 or higher system, $e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0 \quad [\therefore K_a = \infty]$ Thus type 0 and type 1 systems are capable of following a parabolic input. For a type 2 system, the error is finite, but for type 3 or higher systems the error is zero. Again, as increase the type numbers, the error goes on reducing.The terms "position error", "velocity error", "acceleration error" mean steady-state deviations in the output position. A finite velocity error implies that after transients have died out, the input and output move at the same velocity but have a finite position difference.

Steady State Error

The error coefficient Kp, Kv and Ka, describe the ability of a system to reduce or eliminate steady-state error. It is desirable to increase the error coefficients while maintaining the transient response within an acceptable range. Table shows that a type 0 system gives error for all the three types of inputs. A type 2 system gives error due to one type of input only, which is finite. So a type 2 system is better than a type 0 system or a type 1 system from the steady-state error point of view. Higher types of systems are better from the steady-state error point of view but are less stable.

System	Step Input $r(t) = 1$	Ramp Input $r(t) = 1$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0	$\frac{1}{1+K}$	∞	∞
Type 1	0	$\frac{1}{K}$	∞
Type 2	0	0	$\frac{1}{K}$

The nature of the transient response of a system is dependent upon system poles only and not on the type of input. Therefore, we shall analyse the transient response to one of the standard test signals. A step signal is generally used for this purpose.
TIME RESPONSE OF FIRST ORDER SYSTEM TO STEP INPUT:
Let us consider a simple RC circuit as shown in Fig.



Time response analysis

Write the circuit equation as

$$e_i = Ri + \frac{1}{C}\int idt$$
$$and e_o = \frac{1}{C}\int idt$$

Taking Laplace transform of these two equations or,

$$E_i(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

$$E_i(s) = RI(s)\frac{1}{Cs}I(s)$$

and

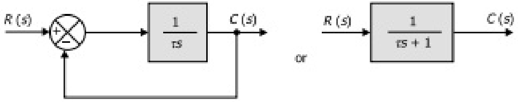
$$\text{Transfer function } = \frac{\text{output}}{\text{input}} = \frac{E_o(s)}{E_i(s)} = \left(\frac{1}{1 + RCs}\right)$$

where $\tau = RC =$ Time constant of RC circuit

$$\therefore$$

$$T.F = \left(\frac{1}{1+\tau s}\right)$$

The block diagram of the system is shown in Fig.



Here

$$G(s) = \frac{1}{\tau s}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Time response analysis of first order system with unit step input

$H(s) = 1$ for unity feedback So,

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{\tau s}}{1 + \frac{1}{\tau s}} = \frac{1}{\tau s + 1}$$

So,

$$C(s) = R(s)\frac{1}{(s\tau + 1)}$$

Time response: Time response to the unit step input, $R(s) = \frac{1}{s}$ will now be calculated: Putting $R(s) = \frac{1}{s}$,

ol,

$$C(s) = \frac{1}{s(\tau s + 1)}$$

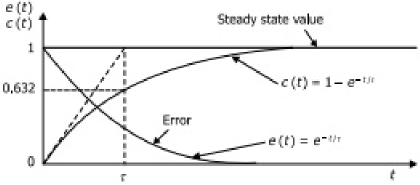
$$C(s) = \frac{1}{s} - \frac{\tau}{\tau s + 1} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$$

Time response analysis of first order system with unit step input

Taking inverse Laplace transform, Therefore,

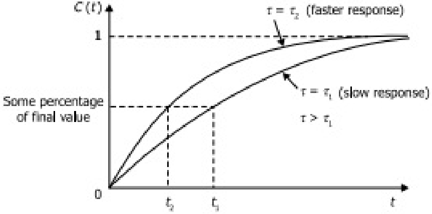
$$C(t) = \left[1 - e^{-ttr}\right]$$

The unit step response of the above system is shown in Fig. As time tends to infinity, the error e(t) goes on reducing and finally becoming zero. The steady state error becomes zero. The time constant is indicative of how fast the system tends to reach the final value. The speed of the response can be quantitatively defined as the time for the output to become a particular percentage of its final value.



Time response analysis of first order system with unit step input

A large time constant corresponds to a sluggish response and a small time constant corresponds to a fast response as shown in Fig. As shown in Fig, time constant t1 is greater than t2 and hence the response is as shown, i.e. it will take more time to reach the final value.

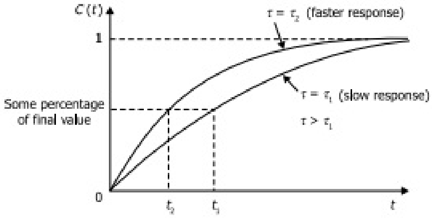


Time response analysis of first order system with unit step input

Error is $e(t) = r(t) - c(t)$
or, $e(t) = 1 - \left(1 - e^{-t/\tau}\right)$
or, $e(t) = e^{-t/\tau}$
Steady-state error is given by

$$e_x = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} e^{-t/r} = 0$$

The error response has been shown in Fig.



Time response analysis of first order system with unit ramp input

The transfer function of a first order system is written as

$$\frac{C(s)}{R(s)} = \frac{1}{\tau s + 1}$$

$$R(s) = \frac{1}{s^2} \text{ (for ramp input)}$$

$$C(s) = \frac{R(s)}{\tau s + 1}$$

or,

$$C(s) = \frac{1}{s^2(\tau s + 1)}$$

or,

$$C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

Time response analysis of first order system with unit ramp input

Taking inverse Laplace transform,

$$C(t) = t - \tau\left(1 - e^{-t/\tau}\right)$$

$$R(s) = \frac{1}{s^2}$$

$$r(t) = t$$

Error is given by

$$e(t) = r(t) - c(t)$$

or

$$e(t) = t - \left[t - \tau\left(1 - e^{-t/\tau}\right)\right]$$

$$e(t) = \tau\left(1 - e^{-t/\tau}\right)$$

Time response analysis of first order system with unit ramp input

Steady-state error is given by

$$e_{\boldsymbol{ss}} = \lim_{t\to\infty} e(t) = \tau$$

Therefore, for a ramp input reducing the system time constant improves the speed of response of the system as well as reduces its steady-state error to a ramp input. We, therefore, need to examine only the steady state error to ramp input which can also be obtained by applying the final value theorem as follows.

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) \\ &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s[R(s) - C(s)] \\ &= \lim_{s\to 0} \left[\frac{1}{s^2} - \frac{1}{s^2(\tau s + 1)}\right] \end{aligned}$$

Time response analysis of first order system with unit ramp input

$$= \lim_{s\to 0} \frac{s}{s^2}\left[1 - \frac{1}{(\tau s + 1)}\right]$$

$$= \lim_{s\to 0} \frac{s}{s^2}\left[1 - \frac{1}{(\tau s + 1)}\right]$$

$$= \lim_{s\to 0} \frac{1}{s}\left[\frac{\tau s + 1 - 1}{\tau s + 1}\right]$$

$$= \lim_{s\to 0} \frac{1}{s}\left[\frac{\tau s}{\tau s + 1}\right]$$

$$= \lim_{s\to 0} \left[\frac{\tau}{\tau s + 1}\right]$$

$$e_{ss} = \tau$$

Time response analysis of first order system with unit impulse input

For unit impulse input $r(t) = \delta(t)$ where $\delta(t)$ is an unit impulse function.
So, $R(s) = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{1}{\tau s + 1}$$

$$C(s) = \frac{1}{\tau s + 1}$$

$$c(t) = \frac{1}{\tau} e^{-t/\tau}$$

Steady-state error

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} sE(s) \\ &= \lim_{s \to 0} s[R(s) - C(s)] \\ &= \lim_{s \to 0} \left[1 - \frac{1}{(\tau s + 1)}\right] \end{aligned}$$

Time response analysis of first order system with unit impulse input

$$= \lim_{s \to 0} s \left[\frac{\tau s + 1 - 1}{\tau s + 1} \right]$$

$$= \lim_{s \to 0} s \left[\frac{\tau s}{\tau s + 1} \right]$$

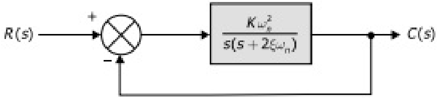
$$e_{ss} = 0.$$

To summarise, the response of a first order system having transfer function $\left(\frac{1}{\tau s+1}\right)$ to different input signals has been tabulated in Table.

Signal	Input $r(t)$	Output $c(t)$	Steady state error, $e_{ss}(t)$
(i) Ramp	t	$t - \tau(1 - e^{-t/\tau})$	τ
(ii) Unit step	1	$1 - e^{-t/\tau}$	0
(iii) Impulse	$u(t) = \delta(t)$	$\frac{1}{\tau} e^{-t/\tau}$	0

Time Response of Second Order Systems

The block diagram representation of the second order system with unity feedback has been shown in Fig.



A second order system is represented by the equation,

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

The transfer function of a second order system is given as,

$$T \cdot F. = \frac{\omega_n^2}{s^2 + 2\xi\omega_s s + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{a_0}{a_2}} \text{ and } \xi = \frac{a_1}{2\sqrt{a_0 a_2}}$$

ω_n is the undamped natural frequency and ξ is the damping ratio.

Time Response of Second Order Systems

The characteristic equation is written by equating the denominator of the transfer function to zero. The characteristic equation of the second order system is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

The roots of the characteristic equation are

$$\begin{aligned} s_1, s_2 &= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \\ &= \sigma \pm j\omega_d \end{aligned}$$

Here s_1, s_2 are the complex frequencies, $\sigma = \xi\omega_n$, is called the attenuation, and $\omega_d = \omega_n\sqrt{1 - \xi^2}$, is called the frequency of damped oscillation.

Time Response of Second Order Systems

So, the characteristic equation of the closed-loop second order system is given by

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

If $\zeta \geq 1$, the roots of the equation are real and the system response will be over damped. On the other hand, if $\zeta < 1$, the roots of the equation are complex conjugate and the system response will be oscillatory in nature. The condition of stability of the system is that the value of ζ should be positive. So for $\zeta \geq 1$, the real roots the of equation are

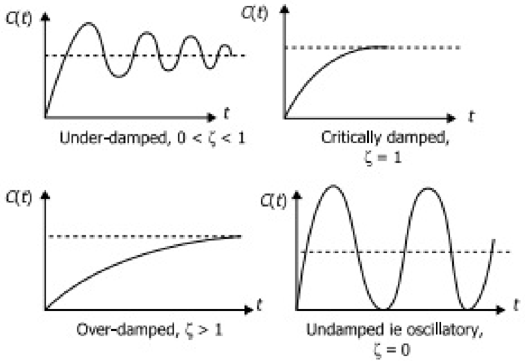
$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

For $\zeta < 1$, the complex conjugate roots of the equation are

$$\begin{aligned} s_1, s_2 &= -\zeta\omega_n \pm j\omega_k\sqrt{1-\zeta^2} \\ &= -\sigma \pm j\omega_d \end{aligned}$$

According to different values of ζ, the time response of a second order control system can be classified as follows.
a) Underdamped: For $0 < \zeta < 1$, the transient response is oscillatory in nature, but decays exponentially to give a stable response.
b) Critically damped: For $\zeta = 1$, the response just becomes non-oscillatory and gives a stable response after transient disappears.
c) Overdamped: For $\zeta > 1$, the response is non-oscillatory and gives a somewhat delayed stable response after transient disappears.
d) Undamped: For $\zeta = 0$, the transient does not disappear and the response gives a sustained oscillation.

Time Response of Second Order Systems



Time Response of Second Order Systems Subjected to Unit Step Input

a) Under-damped case (when $0 < \zeta < 1$) Here $R(s) = \frac{1}{s}$ and so $C(s)$ can be written as

$$C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right.}$$
$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\frac{\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

[where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ = damped natural frequency]
Taking inverse Laplace transform, we get

$$C(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)$$
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

Time Response of Second Order Systems Subjected to Unit Step Input

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left(\sin\phi \cos\omega_d t + \cos\phi \sin\omega_d t\right)$$

$\left[\because \omega_n \sin\phi = \omega_d = \omega_n \sqrt{1 - \zeta^2}\right]$

[where $\cos\phi = \zeta$]

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \phi\right)$$

Thus, time response

$C(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[\left(\omega_n \sqrt{1-\zeta^2}\right) t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right].$

Time Response of Second Order Systems Subjected to Unit Step Input

As $\lim_{t\to\infty} c(t) = 1$, the time response reaches its steady-state value of unity and hence the steady-state error (e_{ss}) becomes zero as deduced below.

$$\begin{aligned}
e_{ss} &= \lim_{t\to\infty} [r(t) - c(t)] \\
&= \lim_{s\to 0} s[R(s) - C(s)] \\
&= \lim_{s\to 0} s\left[\frac{1}{s} - \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right.}\right] \\
&= \lim_{s\to 0} \left[1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_s s + \omega_n^2}\right] \\
&= 0
\end{aligned}$$

Time Response Of second Order Systems Subjected to Unit Step Input

b) Critically damped case (when $\zeta = 1$) As $R(s) = \frac{1}{s}$, and so $C(s)$ can be written as

$$\begin{aligned} C(s) &= \frac{\omega_n^2}{s\left(s^2 + 2\omega_n s + \omega_n^2\right)} \quad [\text{ Putting } \zeta = 1] \\ &= \frac{\left(s + \omega_n\right)^2 - s\left(s + \omega_n\right) - s\omega_n}{s\left(s + \omega_n\right)^2} \\ &= \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{\left(s + \omega_n\right)^2} \end{aligned}$$

Taking inverse Laplace transform on both sides, we get

$$\begin{aligned} c(t) &= 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \\ &= 1 - e^{-\omega_n t}\left(1 + \omega_n t\right) \end{aligned}$$

The unit step response for this case also approaches unity, as $\lim_{t \to \infty} c(t) = 1$ and also steadystate error $e_{ss} = 0$ as before.

Time Response Of second Order Systems Subjected to Unit Step Input

c) Overdamped case (when $\zeta > 1$) Here also $R(s) = \frac{1}{s}$, and $C(s)$ may be written as $C(s) = \frac{\omega_n^2}{s(s^2+2(\omega_n s+\omega_n^2)}$

$$\text{or, } C(s) = \frac{\omega_n^2}{s\left[\left(s+\zeta\omega_s\right)^2 - \omega_n^2\left(\zeta^2-1\right)\right]}$$

$$= \frac{1}{s\left[s+\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right) \mid \left[s+\omega_n\left(\zeta-\sqrt{\zeta^2-1}\right)\right]\right.}$$

$$= \frac{1}{s} + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)\left[s+\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)\right]} \quad \text{(obta}$$

$$- \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta-\sqrt{\zeta^2-1}\right)\left[s+\omega_n\left(\zeta+\sqrt{\zeta^2-1}\right)\right]}$$

Time Response Of second Order Systems Subjected to Unit Step Input

Taking inverse Laplace transform on both sides, we have

$$c(t) = 1 + \frac{e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega t}}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)} - \frac{e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega t}}{2\sqrt{\zeta^2-1}\left(\zeta-\sqrt{\zeta^2-1}\right)}$$

If ζ becomes comparatively larger than 1 , then the second term having a smaller time constant $\left[1/\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n\right]$ decays more quickly than the third term with the larger time constant $\left[1/\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n\right)\right]$.
So after the time of the smaller time constant has elapsed, the response is similar to that of a first order system and $C(s)$ may be approximated as

$$\frac{C(s)}{R(s)} = \frac{1}{1+\tau s} \quad \text{or,} \quad C(s) = \frac{1/s}{1+\tau s}$$

Time Response Of second Order Systems Subjected to Unit Step Input

Substituting value of τ, we get $C(s) = \frac{1/s}{1+\left[1/\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n\right]s}$

$$= \frac{\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n}{s\left[s + \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n\right]}$$

Taking inverse Laplace transform, we get

$$C(t) = 1 - e^{-14-\sqrt{\zeta^2-1}|\Delta t} \quad \text{for } \zeta >> 1$$

Here also

$$\lim_{t\to\infty} C(t) = 1 \text{ and } e_{xx} = 0$$

Time Response Of second Order Systems Subjected to Unit Step Input

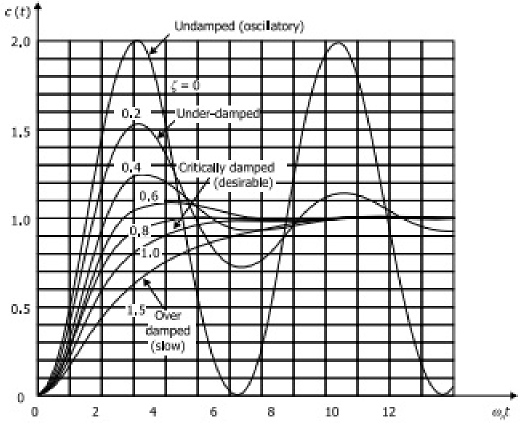
d) Undamped case (when $\zeta = 0$) As and $R(s) = \frac{1}{s}, \zeta = 0$, we may write $C(s)$ as given below.

$$
\begin{aligned}
C(s) &= \frac{\omega_k^2}{s\left(s^2 + \omega_\varepsilon^2\right)} = \frac{\left(s^2 + \omega_n^2\right) - s^2}{s\left(s^2 + \omega_n^2\right)} \\
&= \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}
\end{aligned}
$$

Taking inverse Laplace transform on both sides, we have

$$
C(t) = 1 - \cos \omega t
$$

Time Response Of second Order Systems Subjected to Unit Step Input



The unit step response is easy to generate and mathematically the response to any input can be derived if the response to a step input is known. Therefore, the performance characteristics of a control system are described in terms of transient response to a unit step input; with standard initial conditions of output and all time derivatives being zero when the system is at rest. The time response of second and higher order control systems to a unit step input is generally damped oscillatory in nature before reaching steady state. The following are the transient response specifications (as shown in Fig. 7.13) of a control system to a unit step input.

a) Rise time (t_r) : It is the time required for the response to rise from 10 per cent to 90 per cent (for overdamped or critically damped systems) and 0 per cent to 100 per cent (for underdamped systems) of its final value. The 10 per cent to 90 per cent and 0 per cent to 100 per cent rise time are commonly used for overdamped and underdamped second order systems respectively. b) Peak time (t_p) : The peak time is the time required for the response to reach the first peak of the overshoot. See Fig. 7.13. c) Maximum overshoot and maximum percentage overshoot: The maximum overshoot (M_p) is the maximum peak value of the response measured from unity. So M_p is given by $M_p^{p'} = c\left(t_p\right) - 1$ If the steady-state value is not unity, then the maximum per cent overshoot as defined below is used. Maximum percent overshoot

$$= \frac{c\left(t_p\right) - c(\infty)}{c(\infty)} \times 100$$

d) Setting time (t)): It is the time required for the response curve to reach and stay within a specified tolerance band (either 2 per cent or 5

Transient Response specifications

Assuming the system to be underdamped second order, we shall obtain the rise time (t_r), peak time (t_p), maximum overshoot (M_p) and settling time (t_s) in terms of ζ and ω_n. We have deduced (see equation (7.17)) earlier that time response $c(t)$ under unit step input is given by:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\left(\sqrt{1-\zeta^2}\right)\omega_n t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

a) Rise time (t_r) : Rise time is obtained from $c\,(t_r) = 1$ Putting $c(t) = 1$ in the above equation we get

$$1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 1$$

$$\sin\left(\left(\omega_n\sqrt{1-\zeta^2}\right)t_r + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0 = \sin\pi$$

Now, as $0 < \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} < \frac{\pi}{2}$ for $0 < \zeta < 1$

$$\therefore \left(\omega_n \sqrt{1-\zeta^2}\right) t_r + \tan^{-1} \frac{1-\zeta^2}{\zeta} = \pi$$

$$t_r = \frac{\pi - \tan^{-1} \frac{1-\zeta^2}{\zeta}}{\omega_n \sqrt{1-\zeta^2}} = \text{ The response time}$$

b) Peak time (t_p) : Peak time is obtained by differentiating $c(t)$ with respect to t and equating to zero. At maxima the slope is zero. Therefore peak time is obtained from $\left.\frac{dc(t)}{dt}\right|_{\mathrm{rm}_r} = 0$ or,

$$\frac{\zeta \omega_n e^{-\langle i+2,}}{\sqrt{1-\zeta^2}} \sin \left[\left(\omega_n \sqrt{1-\zeta^2}\right) t_p + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

$$-\omega_n e^{-\langle \Delta_2 f} \cos \left(\left(\omega_n \sqrt{1-\zeta^2}\right) t_p + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right) = 0$$

$$\tan \left(\omega_n \sqrt{1-\zeta^2} t_p + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

The general solution of the above equation is

$$\left(\omega_\kappa\sqrt{1-\zeta^2}\right)t_F = n\pi$$

where $n = 0, 1, 2, 3, \ldots$ and so on. As t_f is the time required for the response to reach the first peak of the overshoot, so t_p is obtained for $n = 1$. (By putting $n = 2$ we can get t_p for the second peak, and so on.) For the peak overshoot, $\left(\omega_n\sqrt{1-\zeta^2}\right)t_p = \pi$ or, $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ = Peak for the first maxima. Here $n = 1$ first undershoot $n = 2$ second overshoot etc. c) Maximum overshoot (M_p) : Maximum overshoot M_p is found by substituting the value of t_p in the expression for $c(t)$ and subtracting the steady state response value from it.

Transient Response specifications

Control Systems

Unit II: ROUTH-HURWITZ CRITERION

Sub-code: 18EE502
Department of Electrical Engineering
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October 22, 2021

Introduction

System stability is one of the most important performance specification of a control system. A system is considered unstable if it does not return to its initial position but continues to oscillate after it is subjected to any change in input or is subjected to undesirable disturbance.

There are many definitions for stability, depending upon the kind of system or the point of view.

A linear, time-invariant system is stable if the natural response approaches zero as time approaches infinity.

A linear, time-invariant system is unstable if the natural response grows without bound as time approaches infinity.

A linear, time-invariant system is marginally stable if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

Thus, the definition of stability implies that only the forced response remains as the natural response approaches zero.

Introduction

When one is looking at the total response, it may be difficult to separate the natural response from the forced response. However, we realize that if the input is bounded and the total response is not approaching infinity as time approaches infinity, then the natural response is obviously not approaching infinity.

Thus, our alternate definition of stability and instability, one that regards the total response are

i) A system is stable if every bounded input yields a bounded output.

ii) If there is no input, the output should tend to be zero, irrespective of any initial conditions.

POLE-ZERO LOCATION AND CONDITIONS FOR STABILITY

The stability of the closed-loop system can be determined by examining the poles of the closed-loop system, that is, by the roots of the characteristic equation. As we know, the nature of time response of a system is related to the location of the roots of characteristic equation in s-plane. For the system to be stable, the roots should have negative real parts. A system will be stable, unstable, or oscillatory depending upon the positions of the roots of the characteristic equation.

CONDITIONS FOR STABILITY

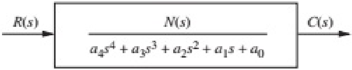
For third and higher order systems, the positiveness of all the coefficients of characteristic equation does not ensure the negativeness of the real parts of complex roots. Therefore, it is a necessary but not sufficient condition for the systems of third and higher order. For example, if you have an unknown parameter in the denominator of a transfer function, it is difficult to determine via a calculator the range of this parameter to yield stability. To examine the sufficient condition for stability, there is a criterion known as the Routh-Hurwitz's stability criterion.

ROUTH'S STABILITY CRITERION AND ITS APPLICATION

The Routh-Hurwitz criterion for stability does not require calculation of the actual values of the roots of the characteristic equation. This criterion tells us about the number of roots on the right side of the imaginary axis. Moreover, this criterion gives just a qualitative result. It is the quickest method if we just want to know whether the system is stable or unstable. The method requires two steps:
(1) Generate a data table called a Routh table
(2) interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $jw - axis$.

ROUTH'S STABILITY CRITERION AND ITS APPLICATION

Look at the equivalent closed-loop transfer function shown in Figure



Begin by labeling the rows with powers of s from the highest power of the denominator of the closed-loop transfer function to s0. Next start with the coefficient of the highest power of s in the denominator and list, horizontally in the first row, every other coefficient. In the second row, list horizontally, starting with the next highest power of s, every coefficient that was skipped in the first row.

TABLE .1 Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			

ROUTH'S STABILITY CRITERION AND ITS APPLICATION

The remaining entries are filled in as follows. Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row. The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right. The table is complete when all of the rows are completed down to s0.

TABLE .2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

ROUTH'S STABILITY CRITERION AND ITS APPLICATION

The Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column. If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.

To generate and interpret a basic Routh table, two special cases can arise. These are

i) The first element in any of the rows of the array is zero, but the others are not.

ii) The elements in one row of the array are all zero.

In the first case, replace the zero element in the first column by an arbitrary small positive number ε, and then proceed with array formation and ultimately let ε tend to zero.

The second case of problem indicates that there are symmetrically located roots in the s-plane. The polynomial whose coefficients are just above the row of zeros in the array is called an acxiliary polynomial. The auxiliary polynomial is always an even polynomial; that is, only even powers of s appear. The roots of the auxiliary equation also satisfy the original equation.

To continue with the array, the following steps should be adopted.

a) Form the auxiliary equation, $A(s) = 0$;

b) Take derivative of the auxiliary equation with respect to s and equate to zero. i.e.,

$$\frac{dA(s)}{ds} = 0$$

c) Replace the row of zeros with the coefficients of

$$\frac{dA(s)}{ds} = 0$$

d) Continue the array in the usual manner with replaced coefficients.

Control Systems

Unit III: Problems on Root locus and Bode Plots

Sub-code: 18EE502
Department of Electrical Engineering
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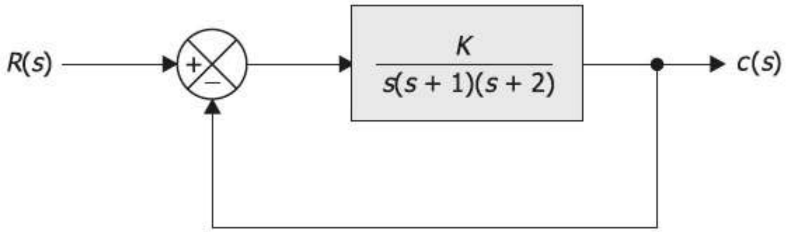
February 13, 2023

Problems1

Prob1: A block diagram representation of a unity feedback control system is shown below. For this system sketch the root locus. Also determine the value of K so that the damping ratio, η of a pair of complex conjugate closed loop poles is 0.5.



Problem1

Solution

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

For determining the open loop poles, we equate the denominator of $G(s)$ to 0 .

$$\therefore \quad s(s+1)(s+2) = 0$$

a) There are three open loop poles at $s = 0, s = -1$, and $s = 2$.
b) We know that the number of root locus asymptotes will be equal to number of open loop poles minus the number of open loop zeros. Here there is no open loop zero.
There will be three branches of the root locus originating respectively at $s = 0$, $s = -1$ and $s = -2$.

Problem1

c) The three branches of the root locus will move towards infinity, as k changes, along the asymptotic lines whose angles with the real axis are

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}; q = 0, 1, 2$$
$$= (2q+1)180^\circ = 60^\circ, 180^\circ, 300^\circ$$

d) The root locus exist on the real axis between $s = 0$, and $s = -1$; and $s = -2$ moving toward ∞.
e) The centroid, $-\sigma_A$ is calculated as

$$-\sigma_A = \frac{\sum \text{ real parts of poles } - \sum \text{ real parts of zeros}}{\text{number of poles } - \text{ number of zeros}}$$

$$= \frac{(-1-2)-0}{3-0} = -1$$

Problem1

f) The break away points on the real axis is found by putting $\frac{dK}{ds} = 0$.
The characteristic equation is

$$s(s+1)(s+2) + K = 0$$
$$K = -s^3 - 3s^2 - 2s$$
$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0$$

i.e., $3s^2 + 6s + 2 = 0$

$$s_1, s_2 = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times 2}}{2 \times 3}$$
$$= -0.43, -1.57$$

Problem1

g) Intersection of the root locus on the imaginary axis is determined as follows. The characteristic equation of the system is

$$s(s+1)(s+2)+K=0$$
$$s^3+3s^2+2s+K=0$$

or, The Routh Array is

$$\begin{array}{ccc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6-K}{3} & 0 \\ s^0 & K & 0 \end{array}$$

We know that the occurrence of a zero row in the Routh array indicates the presence of symmetrically located roots in the s-plane. For this,

$$\frac{6-K}{3}=0$$
$$K=6$$

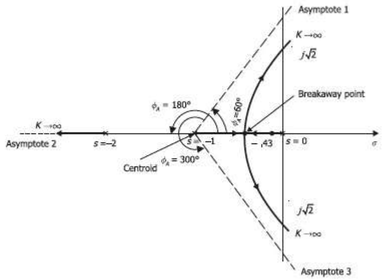
or,

Problem1

The auxiliary equation is or,

$$3s^2 + K = 0$$
$$3s^2 = -K = -6$$
$$s = \pm j\sqrt{2}.$$

$$\text{or,} \quad s = \pm j\sqrt{2}.$$



Problem2

The open-loop transfer function of a feedback control system is given by

$$G(s)H(s) = \frac{K}{s(s+3)\left(s^2+2s+2\right)}.$$

Draw the root locus as K varies from 0 to ∞. Also calculate the value of K for which the system becomes oscillatory.

Solution

Number of open loop poles = 4. They are at $s = 0, -3, -1 \pm j1$. There is no open loop zero. There are four root locus branches originating at the four poles with value of $K = 0$. These branches will terminate at ∞ (i.e. open loop zeros). This is because there are no finite zeros. The four branches will tend to reach infinity with the value of K increasing towards an infinite value along asymptotic path.

Problems2

The angles the asymptotes with the real axis are calculated as

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \text{ at } q = 0, 1, 2, 3$$
$$= \frac{(2q+1)180^\circ}{4-0} = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

The centroid of the asymptotes is calculated as

$$\sigma_A = \frac{\sum(-3-1-1) - \sum(0)}{4-0}$$
$$= -\frac{5}{4} = -1.25$$

Therefore, the asymptotes will be originating at $s = -1.25$ on the real axis.

Problem2

The breakaway point or points are calculated using the characteristic equation, which is $1 + G(s)H(s) = 0$ or, or,

$$\frac{K}{s(2+3)\left(s^2+2s+2\right)} = -1$$
$$K = -s(s+3)\left(s^2+2s+2\right)$$
$$= -\left(s^4+5s^3+8s^2+6s\right)$$

To calculate s, we have to make $\frac{dK}{ds} = 0$

$$\frac{dK}{ds} = -\left(4s^3+15s^2+16s+6\right) = 0$$
$$s = -2.3, -0.725 \pm j0.365$$

The breakaway point must be at $s = -2.3$ as it lies on the real axis. The two root locus branches originating at $s = 0$ and $s = -3$ approach each other on the real axis and breakaway at $s = -2.3$.

Problem2

For calculating the value of K at which the root locus intersects the imaginary axis and the system becomes oscillatory, we start with the characteristic equation and the Routh Array as: Characteristic equation is

$$s(s+2)\left(s^2+2s+2\right)+K=0$$
$$s^4+5s^3+8s^2+6s+K=0$$

or, Applying Routh criterion, For stability, $0<K<8.16$. At a value of $K=8.16$, the two root loci intersects the jw axis. The point of intersection is calculated from the auxiliary equation formed from the coefficients of s^2 row when $K=8.16$ as or,

$$s^2=-\frac{K}{34/5}=-\frac{8.16\times 5}{34}=-\frac{40.8}{34}=-1.21$$

$$\therefore \quad s=\pm j1.1$$

Problem2

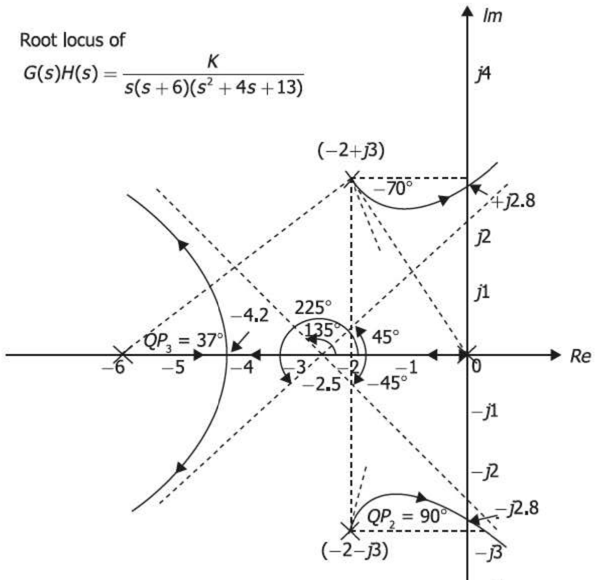
The angle of departure of the two root loci originating at $s = -1 \pm j1$ is calculated as (see Fig. 9.34):

$$\theta_p = 180^\circ - \phi$$

where ϕ is the angle contribution by other poles to this pole.

$$\theta_P = 180^\circ - (90^\circ + 135^\circ + 27^\circ) = -72^\circ$$

Problem2



Root locus of
$G(s)H(s) = \dfrac{K}{s(s+6)(s^2+4s+13)}$
Im
j4
(−2+j3)
−70°
+j2.8
j2
j1
225°
135°
45°
−4.2
QP₃ = 37°
−6
−5
−4
−3
−2
−1
0
Re
−2.5
−45°
−j1
−j2
−j2.8
QP₂ = 90°
(−2−j3)
−j3

Control Systems

Unit III: ROOT LOCUS

Sub-code: 18EE502
Department of Electrical Engineering
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Bapatla

March 16, 2022

Introduction

- Primary aim of a control engineer: To design a control system that meets the desired specifications.
- Stability of the system is also a necessary condition.
- Analytical approach can be followed till second order systems while Routh-Hurwitz criterion can be followed for higher order systems.
- Drawbacks of Routh-Hurwitz criterion -

(a) does not provide sufficient information about relative stability, which may make the system unstable,

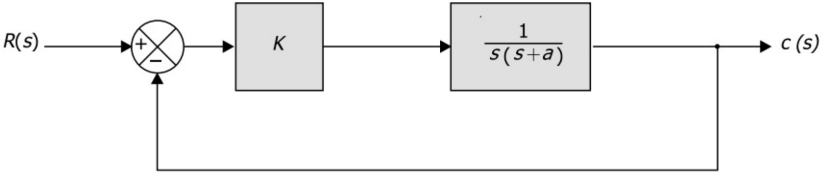
(b) does not help in designing a control system where tuning of parameters is important.

Introduction

- The locus of the migration of the roots, of the characteristic equation, in the s-plane is called 'Root Locus'.
- The root locus technique was introduced by W.R. Evans.
- The root locus technique is a graphical method for sketching the locus of roots of the, characteristic equation in the s-plane as a design parameter of the corresponding system is varied.

THE ROOT LOCUS CONCEPT

- Let us consider a second order system, $G(s) = \frac{1}{s(s+a)}$, and $H(s) = 1$ i.e. the poles of the open loop system is at $s = 0, -a$ and it is an unity feedback system.



- The characteristic equation of the system is

$$s^2 + as + K = 0$$

- The roots of the equation are

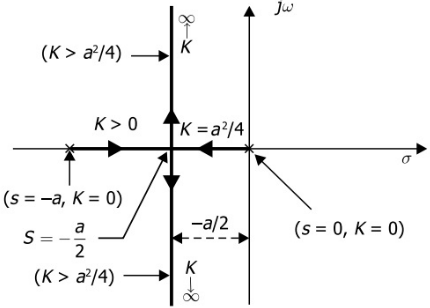
$$s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4K}}{2}$$

- For positive values of a and K, the system is always stable and the roots lie in the left half of the s-plane.
- The relative stability of the system depends on the location of the roots.
- Desired transient response can be obtained by varying the open loop gain K.
- As K is varied from 0 to ∞, the loci of the roots s_1 and s_2 in the s-plane is explained considering three cases:
- Case 1 - For $0 \leq K \leq \frac{a^2}{4}$ the roots are distinct. When $K = 0$, the roots are at $s_1 = 0$ and $s_2 = -a$, which are the poles of the open loop system.
- Case 2 - For $K = \frac{a^2}{4}$, the roots are real and equal i.e. $s_1 = s_2 = -\frac{a}{2}$. As K is varied from 0 to ∞, one root starts moving from $s_1 = 0$ and the other starts moving from $s_2 = -a$ in opposite directions and at $K = \frac{a^2}{4}$ both roots meet at $S = -\frac{a}{2}$

- Case 3 - For $K \geq \frac{a^2}{4}$ the roots are complex and conjugate, given by

$$s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{a^2 - 4K}}{2}$$

- The real parts remain constant and the imaginary part of the roots vary as K varies. Thus, the roots move along the vertical line, one upwards and one downwards.



Advantages

The advantages of root locus technique are:
- Indicating the manner in which the closed loop poles and zeros should be modified so that the response meets system performance specifications.
- Knowledge of the open loop system is sufficient to analyse the behaviour of the system, detailed study of the closed loop system is not required.

System Parameters and Pole Locations

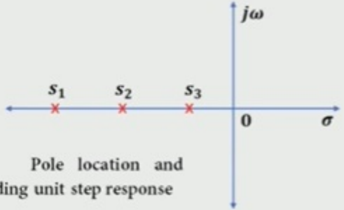
First Order System

- General form of transfer function for a first order system is given by

$$G(s) = \frac{K}{\tau s + 1}$$

  where K and τ are steady state gain and time constant respectively.

- Pole of the first order system is at $s = -\frac{1}{\tau}$; influences the speed of response of the output.



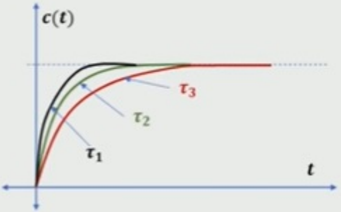


Fig Pole location and corresponding unit step response

Second Order System

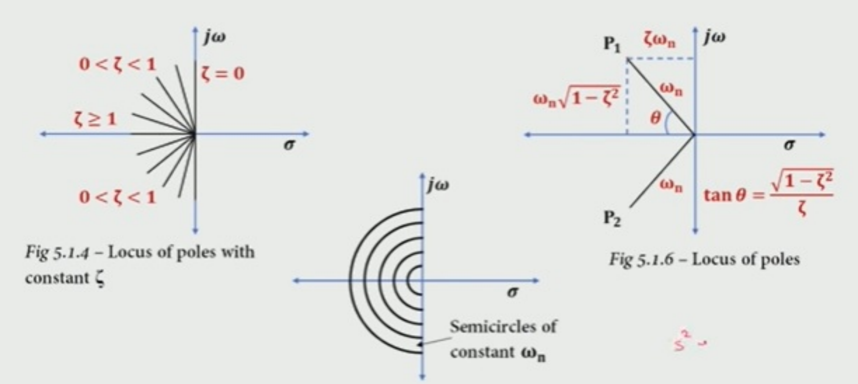
- General form of transfer function for a second order system is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

where ω_n and ζ are natural frequency and damping ratio respectively.

- For $\zeta = 0$, the system is undamped.
- For $0 < \zeta < 1$, the system is under damped.
- For $\zeta = 1$, the system is critically damped.
- For $\zeta > 1$, the system is over damped.

System Parameters and Pole Locations



Evans condition

For higher order systems, this procedure will become complicated and time consuming. Evans developed a simplified graphical technique for root locus plot which is described below. The characteristic equation of the closed-loop system is

$$1 + G(s)H(s) = 0$$

a) Magnitude criterion From equation (9.2), we see that the magnitude of the open-loop transfer function is equal to unity for all the roots of the characteristic equation $|G(s)H(s) = 1|$. b) Angle criterion The angle of the open-loop transfer function is an odd integral multiple of π.

$$\angle G(s)H(s) = \pm 180^\circ (2q + 1)$$

where, q $= 0, 1, 2 \ldots$ The gain factor K does not affect the angle criterion.

Evans condition

The characteristic equation of the system is given by,

$$q(s) = 1 + KG(s)H(s) = 0$$

$$\therefore KG(s)H(s) = P(s) = -1$$

Since 's' is a complex variable equation (5.2.2) can be written in polar form as,

$$|P(s)|\angle P(s) = -1 + j0$$

It is necessary that

$$|P(s)| = 1 \textit{ (magnitude criterion)}$$

$$\angle P(s) = \pm(2q+1)180^{\circ}; \quad q = 0, 1, 2, \ldots \textit{ (angle criterion)}$$

Equation (5.2.4) and (5.2.5) are known as Evans' Conditions.

Evans condition

Consider a system with open loop transfer function in pole-zero form

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

The Evans' conditions for the existence of a point on the root locus are

$$|P(s)| = 1 \textit{ (magnitude criterion)}$$

$$\angle P(s) = \pm(2q+1)180^\circ; \quad q = 0, 1, 2, \ldots \textit{ (angle criterion)}$$

Then applying conditions in equation (5.2.4) and (5.2.5) in equation (5.2.6)

$$K|G(s)H(s)| = \frac{K|s+z_1||s+z_2|\cdots|s+z_m|}{|s+p_1||s+p_2|\cdots|s+p_n|} = \frac{K\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|} = 1$$

and

$$\angle KG(s)H(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{i=1}^{n}\angle(s+p_i) = \pm(2q+1)180^\circ$$

Evans condition

For any point to be on the root locus in the s-plane, it has to satisfy both angle criterion and magnitude criterion. The magnitude criterion is checked after confirming the existence of the point on the root locus by applying the angle criterion. To understand this, let us consider an example where

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Let us examine whether $s = -0.5$ lies on the root locus or not. First we apply the angle criterion as $\angle G(s)H(s)$ at $s = -0.5 = \pm 180^\circ(2q+1)$ where $q = 0, 1, 2, \dots$

$$\begin{aligned}\angle G(s)H(s) &= \frac{K}{(-0.5)(-0.5+1)(-0.5+2)} \\ &= \frac{K}{(-0.5+j0)(0.5+j0)(1.5+j0)} \\ &= \frac{K \lfloor 0^\circ}{180^\circ} = -180^\circ\end{aligned}$$

Evans condition

Since the angle criterion is satisfied, the point $s = -0.5$ lies on the root locus. Now we will also check by applying the magnitude criterion to find the value of K for which the point $s = -0.5$ lies on the root locus. Using magnitude criterion

$$|G(s)H(s) = 1| \text{ at } s = -0.5$$

Here, $\frac{K}{|-0.5||0.5||1.5|} = 1$ o $K = 0.375$ Thus, for $K = 0.375$ point $s = -0.5$ lies on root locus.

ROOT LOCUS CONSTRUCTION RULES

RULE 1:

The root locus is symmetrical about the real axis and the number of branches is equal to the order of the characteristic polynomial (Number of poles of the open loop transfer function).

- The roots of the characteristic equations are either real, imaginary or complex conjugate or combination of all; therefore the root locus is symmetrical about the real axis.
- The root locus above the real axis is mirror image of the root locus below the real axis and vice-versa.
- The number of branches of the root locus is equal to the order of the characteristic polynomial.

RULE 2:

All branches of root locus starts at open loop poles (when $K = 0$) and ends at either open loop zeros or infinity (when $K = \infty$). The number of branches terminating at infinity equals to the difference between the number of poles and number of zeros of $G(s)H(s)$.

- Consider the characteristic equation of an n^{th} order system

$$1 + \frac{K \prod_{i=1}^{m}(s + z_i)}{\prod_{i=1}^{n}(s + p_i)} = 0 \qquad (5.2.15)$$

$$\prod_{i=1}^{n}(s + p_i) + K \prod_{i=1}^{m}(s + z_i) = 0 \qquad (5.2.16)$$

- In equation (5.2.16), all points for which the L.H.S. is zero lie on the root locus.

RULE 2:

- When $K \to \infty$ the first term vanishes and only the second term remains as in equation (5.2.19)

$$\prod_{i=1}^{m}(s + z_i) = 0 \tag{5.2.19}$$

- Equation (5.2.19) shows that all open loop zeros lies on the root locus branches and these open loop zeros are terminating points of the root locus branches.
- In most systems number of open loop poles is greater than the number of open loop zeros i.e. $n > m$. So, only m poles terminate at open loop zeros.
- What happens to the remaining (n-m) poles?

ROOT LOCUS CONSTRUCTION RULES

RULE 2:

- Let us consider $K = 0$. Then equation (5.2.16) is written as

$$\prod_{i=1}^{n}(s + p_i) = 0 \qquad (5.2.17)$$

- The roots of equation (5.2.17) are nothing but the poles of the open loop system. Thus all open loop poles are part of the root locus and all the branches of root locus originate from open loop poles.
- Now equation (5.2.16) is re-written as follows

$$\frac{1}{K}\prod_{i=1}^{n}(s + p_i) + \prod_{i=1}^{m}(s + z_i) = 0 \qquad (5.2.18)$$

ROOT LOCUS CONSTRUCTION RULES

RULE 3:

A point on the real axis lies on the root locus if the sum of the poles and zeros on the real axis to the right of the point is an odd number.

Illustration:

$\angle(s_0+p_1) = \angle(s_0+p_2) = \angle(s_0+z_1) = 180^\circ$

$\angle(s_0+p_3) = \angle(s_0+p_4) = \angle(s_0+z_2) = 0^\circ$

$\angle(s_0+z_3) + \angle(s_0+z_3) = 0^\circ$

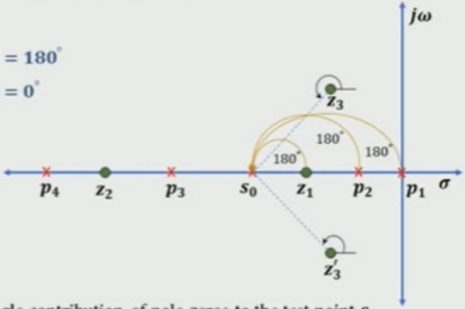


Fig 5.2.3 – Angle contribution of pole-zeros to the test point s_0

RULE 3:

Interpretation:

- Sum of the angles contributed by complex conjugate poles and zeros is zero.
- Angle contribution by every poles and zeros on the real axis to the right of the test point is 180°.
- Angle contribution by every poles and zeros on the real axis to the left of the test point is 0°.
- Therefore, the total angle contribution of the system, whose pole-zero plot is shown in Fig 5.2.3, to the test point s_0 is given by

$$\varphi_{p_1} + \varphi_{p_2} + \varphi_{p_3} + \varphi_{p_4} + \varphi_{z_1} + \varphi_{z_2} + (\varphi_{z_3} + \varphi_{z_3'})$$
$$= -180^\circ - 180^\circ - 0^\circ - 0^\circ + 180^\circ + 0^\circ + 0^\circ = -180^\circ$$

- -180° is odd multiple of -180°. Therefore, s_0 is a point on the root locus.

RULE 3:

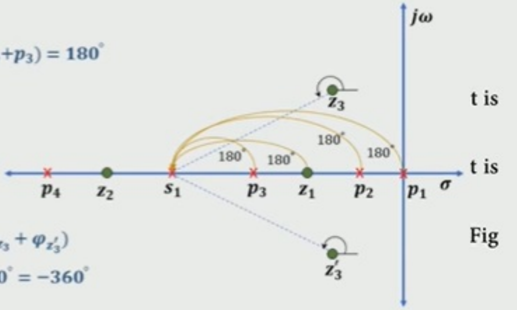
$$\angle(s_0+p_1)=\angle(s_0+p_2)=\angle(s_0+z_1)=\angle(s_0+p_3)=180^\circ$$

$$\angle(s_0+p_4)=\angle(s_0+z_2)=0^\circ$$

$$\angle(s_0+z_3)+\angle(s_0+z_3)=0^\circ$$

$$\varphi_{p_1}+\varphi_{p_2}+\varphi_{p_3}+\varphi_{p_4}+\varphi_{z_1}+\varphi_{z_2}+(\varphi_{z_3}+\varphi_{z_3'})$$

$$=-180^\circ-180^\circ-180^\circ-0^\circ+180^\circ+0^\circ+0^\circ=-360^\circ$$



Example:

Draw the root locus for the unity feedback system with open loop transfer function

$$G(s) = \frac{K(s+1)(s+3)}{s(s+2)(s+4)} \tag{5.2.20}$$

- Number of open loop poles is three. Therefore, number of branches of the root locus is three.
- The three branches of the root locus starts from the open loop poles $s = 0, -2, -4$. Out of the three branches two branches terminate at the open loop zeros and one terminate at infinity.
- All the test points between 0 and -1, between -2 and -3, and between -4 and $-\infty$ lie on the root locus for which the sum of open loop poles and zeros to the right of the test points are 1, 3 and 5 respectively i.e. all have odd no. of poles and zeros to its right.

Example:

RULE 4:

The (n-m) root locus branches that proceed to infinity do so along the asymptotes with angles

$$\varphi_A = \frac{\pm(2q+1)180^\circ}{(n-m)}, q = 0, 1, \cdots (n-m-1) \quad 5.3.1$$

- Consider a test point s_0 at infinity. The angles made by the line joining the test point s_0 and the open loop poles and zeroes are equal to each other (φ_A°).
- Total number of such angles is (n-m). So, the net angle contribution made by all open loop poles and zeroes to the test point s_0 is $-(n-m)\varphi_A^\circ$.
- The total angle contribution at s_0 must satisfy the angle criterion

$$-(n-m)\varphi_A^\circ = \pm(2q+1)180^\circ, q = 0, 1, \cdots (n-m-1) \quad 5.3.2$$

$$\varphi_A = \frac{\pm(2q+1)180^\circ}{(n-m)}, q = 0, 1, \cdots (n-m-1) \quad 5.3.3$$

RULE 5:

The centroid, the point of intersection of the asymptotes with real axis is given by

$$\sigma_A = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{number of poles} - \text{number of zeros}}$$

$n - m$

- Consider the open loop transfer function

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}; m \leq n$$

$$= \frac{K\left[s^m + \left(\sum_{i=1}^{m} z_i\right)s^{m-1} + \cdots + \prod_{i=1}^{m} z_i\right]}{s^n + \left(\sum_{i=1}^{n} p_i\right)s^{n-1} + \cdots + \prod_{i=1}^{n} p_i}$$

- Therefore the characteristic equation is

$$1 + KG(s)H(s) = 1 + \frac{K\left[s^m + \left(\sum_{i=1}^{m} z_i\right)s^{m-1} + \cdots + \prod_{i=1}^{m} z_i\right]}{s^n + \left(\sum_{i=1}^{n} p_i\right)s^{n-1} + \cdots + \prod_{i=1}^{n} p_i} = 0$$

RULE 5:

- Dividing the numerator and denominator by numerator polynomial, we get

$$1+\frac{K}{s^{n-m}+\left(\sum_{i=1}^{n} p_i-\sum_{i=1}^{m} z_i\right)s^{n-m-1}+\cdots+\cdots}=0 \quad 5.3.7$$

- If the test point is selected at infinity, that is for large values of s, the characteristic equation can be approximated to first two terms of the denominator polynomial.

$$1+KG(s)H(s)=1+\frac{K}{s^{n-m}+\left(\sum_{i=1}^{n} p_i-\sum_{i=1}^{m} z_i\right)s^{n-m-1}}=0 \quad 5.3.8$$

- Now let us consider following transfer function which has (n-m) repeated poles at σ_A and no zeros

$$KG(s)=\frac{K}{(s+\sigma_A)^{n-m}} \quad 5.3.9$$

ROOT LOCUS CONSTRUCTION RULES

RULE 5:

- The characteristic equation of the open-loop transfer function in equation (5.3.9) with unity feedback is

$$1 + KG(s)H(s) = 1 + \frac{K}{(s+\sigma_A)^{n-m}} \qquad 5.3.10$$

- The characteristic equation in equation (5.3.10) have (n-m) root locus branches and all originating at σ_A and terminates at infinity.
- The binomial expansion of the characteristic equation in equation (5.3.10) is given by

$$1 + KG(s)H(s) = 1 + \frac{K}{s^{n-m} + (n-m)\sigma_A s^{n-m-1} + \cdots} \qquad 5.3.11$$

- For large values of *s*, equation (5.3.11) is identical to equation (5.3.8).
- Therefore, the straight line root locus branches of the transfer function in equation (5.3.10) are asymptotes of the transfer function (5.3.8) with centroid σ_A

Example:

Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s) = \frac{K}{s(s^2 + 4)} \quad 5.3.16$$

1) The number of open loop poles are three. Therefore, the number of branches in the root locus is three.

2) The three branches of the root locus originate from the open loop poles $s = 0, \pm j2$. All three branches terminate at infinity.

3) All test points on the real axis between 0 and $-\infty$ have odd number of poles to their right hand side. Therefore, all points between 0 and $-\infty$ are part of the root locus.

4) The three root loci that proceed to infinity do so along the asymptotes with angles

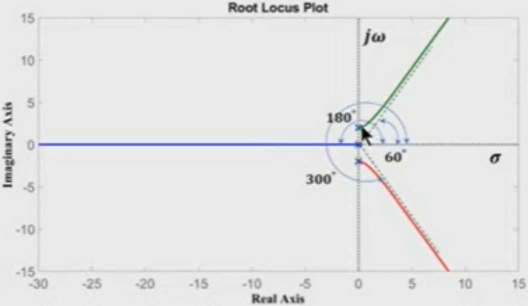
$$\varphi_A = \frac{(2q+1)}{3} 180^\circ, \qquad q = 0, 1, 2 \quad 5.3.17$$

$$\therefore \varphi_A = 60^\circ, 180^\circ, 300^\circ \quad 5.3.18$$

Example:

5) **The centroid, the point of intersection of the asymptotes on the real axis is given by**

$$\sigma_A = \frac{0-0}{3} = 0 \qquad 5.3.19$$



RULE 6:

The break away points (points at which multiple roots of the characteristic equation occur) of the root locus are the solution of $\frac{dK}{ds} = 0$

- Let us assume that the characteristic equation has r multiple roots at $s = s_0$ i.e.

$$1 + KG(s)H(s) = (s + s_0)^r X(s) \qquad 5.3.20$$

where $X(s)$ does not contain the factor $(s + s_0)$.

- Differentiating equation (5.3.20) on both sides w.r.t s yields

$$\frac{d}{ds}[1 + KG(s)H(s)] = r(s + s_0)^{r-1}X(s) + (s + s_0)^r X'(s) \qquad 5.3.21$$

where $X'(s)$ is the derivative of $X(s)$.

- Let the pole-zero form of the characteristic equation be

$$1 + KG(s)H(s) = 1 + \frac{KP(s)}{Q(s)} = 0 \qquad 5.3.22$$

ROOT LOCUS CONSTRUCTION RULES

RULE 6:

- Differentiating equation (5.3.22) on both sides w.r.t 's' we get

$$\frac{d}{ds}[1 + KG(s)H(s)] = \frac{K[Q(s)P'(s) - P(s)Q'(s)]}{Q^2(s)} \quad 5.3.23$$

$$\Rightarrow Q(s)P'(s) - P(s)Q'(s) = 0 \quad 5.3.24$$

- The roots of equation (5.3.22) are the roots of $\frac{d}{ds}[KG(s)H(s)] = 0$
- From equation (5.3.22) we get

$$K = -\frac{Q(s)}{P(s)} \quad 5.3.25$$

- Differentiating w.r.t 's' yields

$$\frac{dK}{ds} = -\frac{P(s)Q'(s) - Q(s)P'(s)}{P^2(s)} = 0 \quad 5.3.26$$

Example:

Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s) = \frac{K(s+1)}{(s+2)(s+3)(s+4)}$$

The open loop poles are at $s = -2, -3, -4$ and the open loop zeros are at $s = -1$. Therefore $n = 3, m = 1$.

1) The number of branches in the root locus are three since $n = 3$.

2) The three branches of root locus originate from open loop poles at $s = -2, -3, -4$ when $K = 0$. Since $m = 1$, out of the three root locus branches only one branch terminates at open loop zero and the remaining two branches terminate at infinity when $K = \infty$.

3) All the points between 0 and -2 and between -3 and -4 lie on the root locus since the sum of poles and zeros to the right of these points is odd (1 and 3 respectively).

Example:

4) The two roots (branches) that proceed to infinity do so along the asymptotes with angles

$$\varphi_A = \frac{(2q+1)}{2}180^{\circ}, \qquad q = 0, 1$$

$$\therefore \varphi_A = 90^{\circ}, 270^{\circ}$$

5) The centroid is given by

$$\sigma_A = \frac{(-2-3-4)-(-1)}{3-1} = -4$$

6) The break away points of the root locus are the solution of $\frac{dK}{ds} = 0$

$$K = -\frac{s^3 + 9s^2 + 26s + 24}{(s+1)}$$

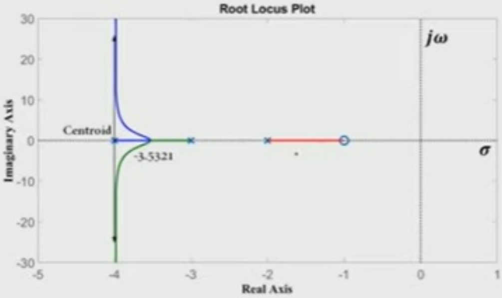
$$\frac{dK}{ds} = -\frac{2s^3 + 12s^2 + 18s + 2}{(s+1)^2} = 0$$

ROOT LOCUS CONSTRUCTION RULES

Example:

$$\Rightarrow s^3 + 6s^2 + 9s + 1 = 0$$

- The roots are $-3.5321, -2.3473, -0.1206$.
- The root -3.5321 alone lies on the root locus. Hence the break away point is at -3.5321.



RULE 7:

The angle of departure from an open loop pole is given by

$$\varphi_p = \pm(2q+1)180^\circ + \varphi, \qquad q = 0, 1, 2, \cdots \qquad 5.4.1$$

where φ is net angle contribution to this pole by all other open loop poles and zeros. Similarly, the angle of arrival at an open loop zero is given by

$$\varphi_z = \pm(2q+1)180^\circ - \varphi, \qquad q = 0, 1, 2, \cdots \qquad 5.4.2$$

- Let p_1, p_2, p_3, p_4, p_5 and p_6 are poles and z_1 is the zero of the system. A point s_0 on the root locus is selected very close to p_5.
- The net angle contributed by all poles and zeros to the pole p_5 is almost equal to the net angle contribution to the arbitrary point s_0 since the pole p_5 and the arbitrary point s_0 are very close to each other.

ROOT LOCUS CONSTRUCTION RULES

RULE 7:

- The net angle contribution of all poles and zeros to this point s_0 is

$$\varphi = \varphi_{z_1} - (\varphi_{p_1} + \varphi_{p_2} + \varphi_{p_3} + \varphi_{p_4} + \varphi_{p_6}) \quad 5.4.3$$

$$\varphi_{net} = \varphi_{p_5} - \varphi \quad 5.4.4$$

where φ is the net angle contributed by all poles and zeros except p_5.

- Again, the net angle contributed by all poles and zeros to a point on the root locus is

$$\varphi_{net} = \pm(2q+1)180^{\circ}, \qquad q = 0, 1, 2, \cdots \quad 5.4.5$$

- Comparing equation (5.3.30) and equation (5.3.31) we get

$$\varphi_{p_5} - \varphi = \pm(2q+1)180^{\circ} \quad 5.4.6$$

$$\Rightarrow \varphi_{p_5} = \pm(2q+1)180^{\circ} + \varphi \quad 5.4.7$$

- Since s_0 is very close to p_5, φ_{p_5} is the angle of departure.

Example:

Sketch the root locus for a unity feedback system with open loop transfer function

$$G(s) = \frac{K(s+1)}{s^2 + 4s + 13}$$

The open loop poles are at $s_{1,2} = -2 \pm j3$ and the open loop zeros are at $s_3 = -1$. Therefore $n = 2, m = 1$.

1) The number of branches in the root locus are two since $n = 2$.

2) The two branches of root locus originate from open loop poles at $-2 \pm j3$ when $K = 0$. Since $m = 1$, out of the two root locus branches only one branch terminates at open loop zero and the remaining one branch terminate at infinity when $K = \infty$.

3) All the points between $-\infty$ and -1 lie on the root locus since the sum of poles and zeros to the right is odd.

Example:

4) The root locus that proceed to infinity do so along the asymptote with angle

$$\varphi_A = \frac{(2q+1)}{1} 180^\circ, \qquad q = 0$$

$$\therefore \varphi_A = 180^\circ$$

5) The break away points of the root locus are the solution of $\frac{dK}{ds} = 0$

$$K = -\frac{s^2 + 4s + 13}{(s+1)}$$

$$\frac{dK}{ds} = -\frac{s^2 + 2s - 9}{(s+1)^2} = 0$$

$$\Rightarrow s^3 + 6s^2 + 9s + 1 = 0$$

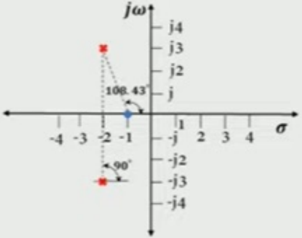
- The roots are -4.16 and 2.16. The break away point is only at -4.16 since the root lie on the root locus and the other root does not.

Example:

6) The angle of departure is given by

$$\varphi_p = \pm 180^\circ + \varphi$$

By calculation $\varphi_{-2+j3} = 198.43^\circ$ **and** $\varphi_{-2-j3} = 161.57^\circ$



ROOT LOCUS CONSTRUCTION RULES

RULE 8 and 9:

- The intersection of root locus with imaginary axis can be determined using the Routh Criterion.
- The open loop gain K at any point s_0 on the root locus is given by

$$K = \frac{\prod_{i=1}^{n} |s_0 + p_i|}{\prod_{i=1}^{m} |s_0 + z_i|}$$

ROOT LOCUS CONSTRUCTION RULES

As mentioned earlier, root locus is the path of the roots of the characteristic equation, $1 + G(s)H(s) = 0$ traced out in s-plane as the system parameter (gain K) is changed.
The root locus diagram or plot can be completed using the following procedure. The procedure is presented in the form of certain rules.
a) Starting and termination of root locus-From the open-loop transfer function, locate the poles and zeros. Each branch of the root locus originates from an openloop pole with $K = 0$ and terminates either on an open-loop zero or at infinity as the value of K increases from 0 to ∞.
In most cases, we will have more poles than zeros. If we have n poles and m zeros, and $n > m$. then $n - m$ branches of the root locus will reach infinity. Because the root loci originate at the poles, the number of root loci is equal to number of poles. b) Root locus on the real axis-The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros. Let the open-loop transfer function of a control system be $G(s) = K(s+1)/(s+2)$. The pole is at $s = -2$ and the zero is at $s = -1$ as shown in Fig. 9.4 (a). The root

d) The member of asymptotes and their angles with the real axis-The $(n-m)$ branches of root loci move towards infinity. They do so along straight line asymptotes. The angle of asymptotes with respect to the real axis is given by

$$\phi_A = \frac{(2q+1)}{n-m}180^\circ, \quad q = 0, 1, 2, \ldots$$

where n is the number of poles and m is the number of zeros. e) Centroid of the asymptotes-The linear asymptotes are centred at a point on the real axis. This is called the centroid which is given by the relation,

$$\sigma_A = \frac{\Sigma \text{ real parts of poles } - \Sigma \text{ real parts of zeros}}{n-m}$$

f) Breakaway points-The root locus breakaway from the real axis where a number of roots are available, normally, where two roots exist. The method of determining the breakaway point is to rearrange the characteristic equation in terms of K. We then evaluate $dK/ds = 0$ in order to find the breakaway point. Since the characteristic equation can have

Control Systems

Unit III: Frequency Analysis

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Department of Electrical Engineering
Bapatla Engineering College
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November 19, 2021

BODE PLOT

Bodeplot is one of the powerful graphical methods of analyzing and designing controlsystems. Introduction of logarithmic plots simplifies the determination of graphical representation of the frequency response of the system. Such logarithmic plots are popularly called Bode plots in honour of H.W. Bode. In Bode plot, we plot logarithm of magnitude versus frequency; and phase angle versus frequency. In this form of representation of a sinusoidal transfer function, the magnitude $G(j\omega)$ in dB which is $20\log|G(j\omega)|$ is plotted against $\log\omega$.

The transfer function of a system in the frequency domain is expressed as

$$G(j\omega) = |G(j\omega)|\angle\phi(\omega)$$

By expressing the magnitude in terms of logarithm to the base 10 , we can write Logarithmic gain $= 20\log_{10} G(j\omega) \mid$ where the units are in decibels.

For a Bode diagram logarithmic gain in dB versus ω and phase angle $\phi(\omega)$ versus ω are plotted separately on the semi-log paper.

BODE PLOT

Bode plots cover a wide range of frequencies due to log scale on x-axis. Due to log scale on y-axis Bode plots cover a wide range of gains. This is the advantage of using log scale. We will now consider a generalized transfer function and explain the Bode plot in details. Let,

$$G(s)H(s) = \frac{K\left[(1+s\tau_1)(1+s\tau_2)(1+s\tau_s)\ldots\ldots\ldots\ldots\ldots\right.}{s^N\left[(1+s\tau_a)(1+s\tau_b)\ldots\ldots.(s^2+2\xi\omega_s s+\omega_s^2)\right]}$$

The sinusoidal form of the transfer function is obtained by substituting s for $j\omega$.

$$G(j\omega)H(j\omega) = \frac{K\left[(1+j\omega T_1)(1+j\omega\tau_2)(1+j\omega\tau_5)\ldots\ldots\ldots j_n^2\right.}{(j\omega)^N\left[(1+j\omega T_a)(1+j\omega T_b)\ldots\ldots\ldots\{(\omega_n^2-\omega^2)+j2\xi\omega_n\right.}$$

$$= \frac{K\left[(1+j\omega T_1)(1+j\omega\tau_2)(1+j\omega\tau_5)\ldots\ldots\ldots\omega_n^2\right.}{(j\omega)^N\left[(1+j\omega\tau_a)(1+j\omega\tau_b)\ldots\ldots\ldots\{(\omega_s^2-\omega^2)+j2\xi\omega_n\omega\right.}$$

Procedure for drawing Bode plot

Now we will explain the procedure for drawing the Bode plot as follows. Magnitude of $G(j\omega)H(j\omega)$ in decibel is given as:

$$\begin{aligned}20\log_{10}|G(j\omega)H(j\omega)| &= 20\log_{10}K + 20\log_{10}|(1+j\omega\tau_1)| \\ &+ 20\log_{10}|(1+j\omega\tau_2)| + 20\log_{10}|(1+j\omega\tau_3)| + \ldots\ldots \\ &- 20N\log_{10}|(j\omega)| - \emptyset\log_{10}|1+j\omega\tau_a| \\ &- 20\log_{10}|1+j\omega\tau_b|\ldots\ldots\ldots\ldots \\ &- 20\log_{10}\left|1-\left(\frac{\omega}{\omega_s}\right)^2 + j2\xi\frac{\omega}{\omega_s}\right|\end{aligned}$$

Procedure for drawing Bode plot

For the phase angle we write,

$$\lfloor G(j\omega)H(j\omega) = \tan^{-1}(0) + \tan^{-1}(\omega\tau_1) + \tan^{-1}(\omega\tau_2)\ldots\ldots\ldots\ldots$$
$$- N\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\omega\tau_a - \tan^{-1}(\omega\tau_b)\ldots\ldots\ldots - \tan^{-1}\left[\frac{2\xi\omega_\varepsilon\omega}{(\omega_n^2-\omega^2)}\right]$$

or,

$$\lfloor G(j\omega)H(j\omega) = \tan^{-1}(\omega\tau_1) + \tan^{-1}(\omega\tau_2)\ldots\ldots..$$
$$-N\times 90 - \tan^{-1}\omega\tau_a - \tan^{-1}(\omega\tau_b) - \tan^{-1}\left[\frac{2\xi\omega_{\mathrm{n}}\omega'}{(\omega_n^2-\omega^2)}\right]$$

Bode plot is a graph obtained from the above equations consisting of two parts as follows. i) Plot of magnitude of $G(j\omega)H(j\omega)$ in decibel versus $\log_{10}\omega$ and ii) Plot of phase angle of $G(j\omega)H(j\omega)$ versus $\log_{10}\omega$.

Methods of Drawing Bode Plot

For plotting magnitude $\mid G(j\omega))H(j\omega) \mid$ in docibels versus $\log_{10}\omega$, we have to add the plots of all the individual factors as included in $G(j\omega)H(j\omega)$ magnitude equation which are listed below.

a) Plot of gain K which is a constant;

b) Plot of poles at the origin, $\left(\frac{1}{j\omega}\right)^N$;

c) Plot of poles on real axis, $\frac{1}{1+j\omega\tau}$

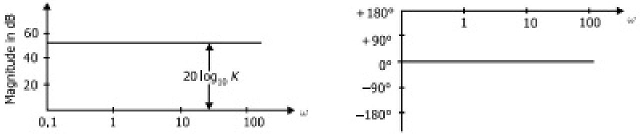
d) Plot of zeros on real axis, $(1 + j\omega\tau)$;

f) Plot of complex conjugate zeros, if present.

Now we will explain the Bode plots of the above individual factors.
a) Plot for the constant gain K.
The magnitude in decibel of the term K is given as

$$K(dB) = 20\log_{10}(K)$$

The above Equation indicates that the magnitude is independent of $\log_{10}\omega$. Assuming K as positive and real, the Bode plot has been drawn as shown in Fig. (a). The phase angle is always zero for any value of ω as shown in Fig. (b).



(a) (b)

b) Plot of the term $\frac{1}{(j\omega)^N}$ representing a pole at the origin
The magnitude of the term $\frac{1}{(j\omega)^N}$ in decibel is written as

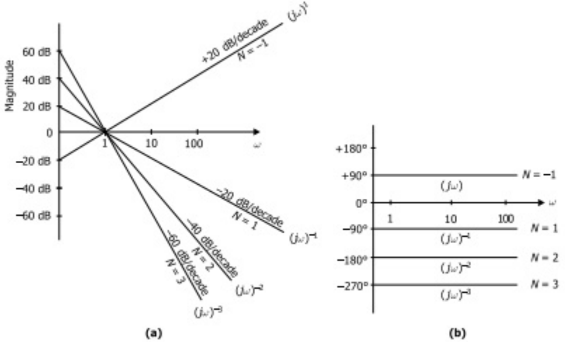
$$20\log_{\mathrm{ta}}\left|\frac{1}{(j\omega)^N}\right| = -20N\log_{\mathrm{se}}(\omega)$$

The magnitude curve will have a slope of -20 dB/ decade for a pole $(N = 1)$
For $N = 2$ the slope will be -40 dB/ decade.
For $N = 3$ the slope will be -60 dB/ decade.

The phase angle is given by

$$\frac{1}{(j\omega)^N} = -N\tan^{-1}\left(\frac{\omega}{0}\right) = -N\tan^{-1}\infty = -N\times 90^\circ$$

Therefore, for a zero at the origin we have a logarithmic magnitude $+20\log_{10}\omega$ where the slope is +20 dB/ decade; and the phase angle is, $\phi(\omega) = 90^\circ$. The graphs have been shown in Fig.



c) Plot for the term $\frac{1}{1+j\omega\tau}$, i.e. for poles on the real axis The magnitude is written as

$$20\log_{10}\left|\frac{1}{(1+j\omega\tau)}\right| = 20\log_{10}\left(\frac{1}{\sqrt{1+\omega^2\tau^2}}\right)$$
$$= -20\log_{10}\left(1+\omega^2\tau^2\right)^{1/2}$$

Let us calculate the magnitude at very low and very high frequencies as when $\omega << \frac{1}{\tau}$, magnitude is

$$20\log_{10}\left|\frac{1}{(1+j\omega\tau)}\right| = -20\log_{\infty} 1 = 0 \text{ dB}$$

when $\omega >> \frac{1}{\tau}$

$$\begin{aligned} \text{magnitude} &= 20\log_{10}\left|\frac{1}{(1+j\omega\tau)}\right| = -20\log_{10}(\omega\tau) \\ &= -20\log_{10}(\omega) - 20\log_{10}(\tau) \end{aligned}$$

Equation is similar to the equation of a straight line $y = mx + c$.
Here,

$$m(\text{slope}) = -20 \text{ dB/decade}$$

$$c = -20\log_{10}(\tau) = 20\log_{10}\left(\frac{1}{\tau}\right)$$

Methods of Drawing Bode Plot

Equations when $\omega << \frac{1}{\tau}$ and $\omega >> \frac{1}{\tau}$ are two curves. To determine where the two curves are intersecting on the 0 dB axis we have to equate equation of when $\omega >> \frac{1}{\tau}$ to zero. Thus, we write or,

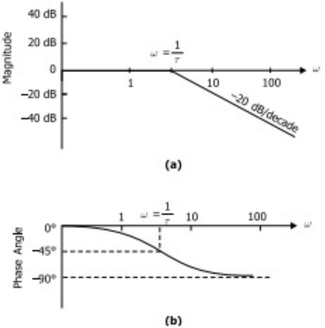
$$0 = -20\log_{10}(\omega) - 20\log_{10}(\tau)$$

$$20\log_{10}(\omega) = -20\log_{10}(\tau) = 20\log_{10}\left(\frac{1}{\tau}\right)$$

$$\omega = \frac{1}{\tau}$$

Hence the two curves intersect on 0 dB axis at $\omega = \omega_c = \frac{1}{\tau}$ where ω_c is called the break frequency or corner frequency.

The phase angle, $\phi(\omega) = -\tan^{-1}(\omega\tau)$
At very low frequency, $\phi(\omega) = -\tan^{-1}(0) = 0^\circ$
At very high frequency, $\phi(\omega) = -\tan^{-1}(\infty) = -90^\circ$
At corner frequency, $\omega = \omega_c = \frac{1}{\tau}$; $\phi(\omega) = -\tan^{-1}\left(\frac{1}{\tau} \times \tau\right) = -\tan^{-1}(1) = -45^\circ$
The Bode diagram for the pole factor $\frac{1}{1+j\omega t}$ has been shown in Fig.



d) Plot of zeros on real axis, $(1 + j\omega\tau)$;
The Bode diagram for a zero factor (factor involving zero in the transfer function) (1+ $j\omega r$) is determined in a similar way as poles on real axis

$$\text{magnitude, } 20 \log_{10} |(1 + j\omega\tau)| = 20 \log_{10} \left(1 + \omega^2\tau^2\right)^{\frac{1}{2}}$$

when $\omega\tau << 1, \omega\tau$ is negligible as compared to 1 therefore, $20 \log_{10} |(1 + j\omega\tau)| = 20 \log_{10}(1) = 0$ dB
And when $\omega T >> 1, \omega\tau$ is much higher than 1 Therefore, $20 \log_{10} |(1 + j\omega\tau)| = 20 \log_{10} \omega\tau = 20 \log_{10}(\omega) + 20 \log_{10}(\tau)$ Equation is similar to a general straight line equation Here,

$$y = mx + c$$

$$m(\text{ slope }) = 20 \text{ dB/ decade}$$

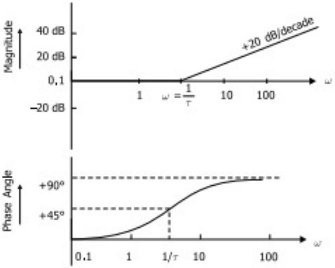
$$C = 20 \log_{10}(\tau)$$

Equations when $\omega\tau << 1 and \omega\tau >> 1$ represent two graphs. The graph of equation lies on 0 dB axis when $\omega\tau << 1$, whereas graph of equation has a slope of 20 dB/ decade when $\omega\tau >> 1$. The intersection of the two graphs is found by equating equation of when $\omega\tau >> 1$ to zero. Thus,

$$0 = 20\log_{10}(\omega) + 20\log_{10}(\tau)$$
$$\omega = \frac{1}{\tau} = \omega_c$$

where ω_c is called the corner frequency.

The Bode diagram for a zero factor of $(1 + j\omega\tau)$ has been shown in Fig.

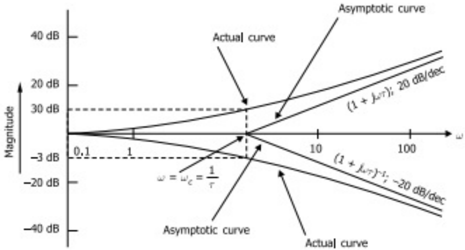


The phase angle has been calculated as

$$\begin{aligned} \text{at } \omega\tau << 1, \quad & \phi = \tan^{-1}(\omega\tau) \\ \text{at } \omega\tau >> 1, \quad & \phi = \tan^{-1}(0) = 0 \\ & \phi = \tan^{-1}(\infty) = 90^\circ \end{aligned}$$

$$\text{at } \omega = \omega_c = \frac{1}{\tau}, \quad \phi = \tan^{-1}\left(\frac{1}{\tau} \times \tau\right) = 45^\circ$$

The magnitude versus $\log_{10}\omega$ graphs for both $(1+j\omega\tau)^{-1}$ and $(1+j\omega\tau)$ have been shown together in Fig.



The exact or actual curves and the asymptotic curves differ from each other to some extent. They are matched at the corner frequency $\omega_c = \frac{1}{\tau}$ as shown in Fig. The maximum error between the exact plot and the asymptotic plot occurs at the corner frequency.

From exact plot the magnitude at $\omega = \omega_c = \frac{1}{\tau}$ is calculate as

$$20\log_{10}|(1+j\omega\tau)| = 20\log_{10}\left|\left(1+j\frac{1}{\tau}\times\tau\right)\right|$$
$$= 20\log_{10}\sqrt{2} = 3 \text{ dB}$$

For the curve for $(1+j\omega\tau)^{-1}$ the magnitude of exact plot at corner frequency will be -3 dB
If the poles or zeros on the real axis are of the form $(1+j\omega\tau)^{\pm N}$, the error in magnitude can be calculated to be equal to $\pm 3N$ dB and the phase angle error will be $\pm N45°$.

Control Systems

Unit III: Frequency Analysis

Sub-code: 18EE502
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November 17, 2021

Introduction

Frequency response methods, developed by Nyquist and Bode in the 1930s, are older than the root locus method, which was discovered by Evans in 1948 (Nyquist, 1932, Bode, 1945).
- The frequency response of a system is defined as the steady state response of the system to a sinusoidal input signal
- The sinusoid is a unique input signal and the resulting output signal for a linear system, is sinusoidal at the steady state.
- It differs from the input waveform only in amplitude and phase angle.
Analysis involves examining the transfer function $G(s)$ when $s = j\omega$ and graphically displaying $G(j\omega)$ as ω varies.

Introduction

- The frequency response of a system is defined as the steady state response of the system to a sinusoidal input signal
- The sinusoid is a unique input signal and the resulting output signal for a linear system, is sinusoidal at the steady state.
- It differs from the input waveform only in amplitude and phase angle.

Advantages

- The frequency response of a LTI system is independent of the amplitude and phase of the input test signal.
- The design and parameter adjustment of the closed loop system for specified closed loop performance can be carried out easily in frequency domain than in time domain.
- The effect of noise disturbance and parameter variations in frequency domain can be easily visualized.
- There is correlation between time domain and frequency domain specifications, so time domain performance of a linear system can be easily predicted using frequency domain specifications.
- Transfer function of the system can be obtained from the frequency response.

Frequency analysis

As mentioned, for analysis of many systems frequency response is of importance since most of the input signals are either sinusoidal or composed of sinusoidal components (harmonics). The starting point for frequency response analysis is the determination of system transfer function. Then in the frequency response the transfer function is expressed in terms of magnitude and phase angle as $M(s) = M \mid \phi$ Let us, consider the transfer function of a first order system as an example.

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{1+s\tau} \quad \text{or} \quad C(s) = \frac{1}{1+s\tau}R(s)$$

For frequency response analysis we replace s by $j\omega$. Therefore,

$$G(j\omega) = \frac{1}{1+j\omega\tau}$$

The system is to be subjected to sinusoidal input and therefore, we take $R(s) = A_i \sin \omega t$ From equation, the output $C(j\omega) = \frac{A_i \sin \omega t}{1+j\omega\tau}$ Magnitude of output, $A_0 = |C(j\omega)| = \frac{A_i}{\sqrt{1+\omega^2\tau^2}}$ Magnitude of input $= A_i$

The dimensionless ratio of output to input is given as

$$M = \left|\frac{A_0}{A_i}\right| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

and the phase angle

$$\phi = \tan^{-1}(-\omega\tau) = -\tan^{-1}\omega\tau$$

$$\text{Time lag } = \frac{1}{\omega}\tan^{-1}\omega\tau$$

Now let us plot M versus ω and ϕ versus ω. When

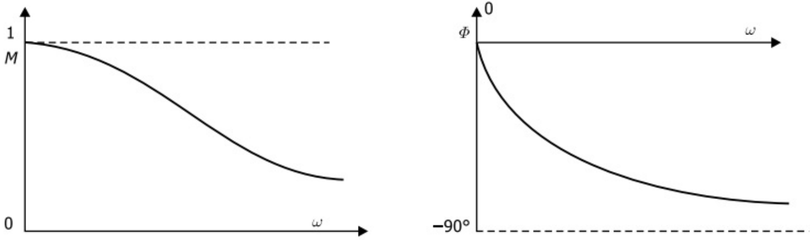
$$\omega = 0, M = \frac{A_0}{A_i} = 1 \text{ and } \phi = 0$$

When

$$\omega = \infty, M = \frac{A_0}{A_i} = 0 \text{ and } \phi = -90^\circ$$

Frequency analysis

As ω increases from 0 to ∞, the magnitude gradually decreases from 1 to 0 and the angle of lag, ϕ increases from 0 to $-90°$. Thus higher the frequency, higher is the attenuation (decay) of the output and greater is the angle of lag between output and input. The frequency response characteristic of a first order system has been shown in Fig.



frequency response specifications

Resonant Peak (M_r)

- The maximum value M_r of magnitude $|M(j\omega)|$ is known as the resonant peak. A system with large resonant peak will exhibit a large overshoot.
- In general, the magnitude of M_r gives indication on the relative stability of a stable closed-loop system.

Resonant Frequency (ω_r)

- The frequency at which the output of the system has maximum magnitude is resonant frequency.

Bandwidth (ω_b)

- The bandwidth BW is the frequency at which $|M(j\omega)|$ drops to 70.7% of, or 3dB down from, its zero-frequency value.

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

For comparative study of time response and frequency response of a system, let us consider a second-order system. We had seen earlier that the transfer function of a second-order system can be expressed as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ζ is the damping ratio and $\boldsymbol{\omega}_n$ is the undamped natural frequency of oscillations. For the sinusoidal transfer function, we will put $s = j\omega$ in the above expression.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega}$$

Dividing by ω_n^2

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}}$$

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

Let $\frac{\omega}{\omega_n} = u$, then

$$M(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1-u^2) + j2\zeta u} = M\angle\phi$$

$$|M(j\omega)| = M = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

$$| \, M(j\omega) = \tan^{-1}\frac{2\zeta u}{1-u^2}$$

From equations (10.4) and (10.5), it is seen that if $u = 0, M = 1$ and $\phi = 0$ if $u = 1, M = \frac{1}{2\zeta}$ and $\phi = -\frac{\pi}{2}$ if $u \to \infty, M \to 0$ and $\phi = -\pi$

As u changes from 0 to ∞, M changes from 1 to 0 and ϕ changes from 0 to $-\pi$. The frequency at which M has maximum value is called the resonant frequency ω_r. Let, $u_r = \omega_J/\omega_n$ where u_r is called the normalized resonant frequency. We will differentiate M with respect to u and substitute $u = u_r$ and then equate to zero.

$$\frac{d}{du}\left[\left(1-u_r^2\right)^2+\left(2\zeta u_r\right)^2\right]^{-1/2}=0$$

or

$$-\frac{1}{2}\left[\left(1-u^2\right)^2+\left(2\zeta u\right)^2\right]^{-3/2}\left[-4u+4u^3+8\zeta^2 u\right]=0$$

or

$$-\frac{4u^3-4u+8c^2u}{\frac{1}{2}\left[(1-u^2)+(2\zeta u)^2\right]^{3/2}}=0$$

Substituting $u = u_r$,

$$4u_r^3-4u_r+8\zeta^2u_r=0$$

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

From equations (10.4) and (10.6), we have

$$M_r = \frac{1}{\sqrt{\left(1 - u_r^2\right)^2 + \left(2\zeta u_r\right)^2}}$$

$$M_r = \frac{1}{\sqrt{\left[1 - \left(1 - 2\zeta^2\right)\right]^2 + \left[4\zeta^2\left(1 - 2\zeta^2\right)\right]}}$$

$$M_r = \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}}$$

$$M_r = \frac{1}{\sqrt{4\zeta^2 - 4\zeta^4}} = \frac{1}{\sqrt{4\zeta^2\left(1 - \zeta^2\right)}}$$

$$M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

From equation (10.5),

$$\angle M(j\omega) = \phi = \tan^{-1}\frac{2\zeta u_r}{1-_r^2} = \tan^{-1}\frac{2\zeta\sqrt{1-2\zeta^2}}{1-1+2\zeta^2}$$

or

$$\phi = \tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta}$$

The characteristics of magnitude M and phase angle ϕ for normalized frequency u for some value of ζ have been shown in Fig. 10.5. 10.3.1 Correlation Between Time Domain and Frequency Domain Parameters We had calculated time domain specification parameters like maximum overshoot M_p, peak time t_p, rise time t_p, settling time t_s, etc. M_p is calculated as

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

and in the frequency domain resonant peak M_r has been calculated as

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

It is observed that both M_p and M_r are the functions of damping ratio ζ. As ζ is increased, the value of maximum overshoot M_p goes on decreasing. When ζ is made equal to 1 , the overshoot disappears, that is, no overshoot is produced by the system response. In the frequency domain, resonant peak M_r will disappear when $\zeta > 1/\sqrt{2}$, that is, $\zeta > 0.707$ For lower value of ζ both M_p and M_r will be large which is undesirable. In practice, the value of ζ is kept such that both the performance indices, that is, M_p and M_r, are correlated. Therefore, ζ is generally designed as $0.4 < \zeta < 0.707$.

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

We had resonant frequency ω_r and damped frequency of oscillation ω_d as

$$\omega_r = \omega_n\sqrt{(1-2\zeta^2)}$$
$$\omega_d = \omega_n\sqrt{(1-\zeta^2)}$$

By comparing these two values, it can be said that there exists correlation between resonant frequency and damped frequency of oscillations. The ratio of

$$\frac{\omega_r}{\omega_d} = \sqrt{\frac{(1-2\zeta^2)}{(1-\zeta^2)}}$$

is also a function of ζ. When ζ lies between 0.4 and 0.707, both ω_d and ω_r are comparable to each other.

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

Bandwidth is the range of frequencies over which the value of M is equal to or greater than $1/\sqrt{2}$, that is, 0.707.

$$M = \frac{1}{\sqrt{\left(1-u^2\right)^2 + (2\zeta u)^2}}$$

Let u_b = normalized bandwidth $= \frac{\omega_b}{\omega_n}$ Then,

$$M = \frac{1}{\sqrt{\left(1-u_b^2\right)^2 + \left(2\zeta u_b\right)^2}} = \frac{1}{\sqrt{2}}$$

or

$$\left(1-u_b^2\right)^2 + \left(2\zeta u_b\right)^2 = 2$$

or

$$1 + u_b^4 - 2u_b^2 + 4\zeta^2 u_b^2 = 2$$

or

$$u_b^4 - 2u_b^2 + 4\zeta^2 u_b^2 + 1 = 2$$

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

Let $u_b^2 = x$, then

$$x^2 - 2x + 4\zeta^2 x + 1 = 2$$

or

$$x^2 - 2x\left(1 - 2\zeta^2\right) - 1 = 0$$

$$x = \frac{2\left(1 - 2\zeta^2\right) \pm \sqrt{4\left(1 - 2\zeta^2\right)^2 + 4}}{2}$$

or

$$x = 1 - 2\zeta^2 \pm \sqrt{1 - 4\zeta^2 + 4\zeta^4 + 1}$$

CORRELATION BETWEEN TIME RESPONSE AND FREQUENCY RESPONSE

or

$$x = 1 - 2\zeta^2 \pm \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

or

$$u_b^2 = 1 - 2\zeta^2 \pm \sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

or

$$u_b = \sqrt{1 - 2\zeta^2 \pm \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

As $u_b = \frac{\omega_b}{\omega_n}$

$$\omega_b = u_b w_n = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

ω_b is the denormalized bandwidth.

Control Systems

Unit III: Problems

Sub-code: 18EE502
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December 24, 2021

Sketch the polar plot of a system whose transfer function is given as

$$G(s)=\frac{10}{s(s+1)(s+2)}$$

Solution

By putting $s=j\omega$ in the transfer function,

$$G(j\omega)=\frac{10}{j\omega(j\omega+1)(j\omega+2)}$$
$$=\frac{10}{-3\omega^2+j(2\omega-\omega^3)}$$
$$=\frac{10\left[-3\omega^2-j(2\omega-\omega^3)\right]}{9\omega^4+(2\omega-\omega^3)^2}$$
$$=\frac{-30\omega^2}{9\omega^4+(2\omega-\omega^3)^2}+j\frac{10(\omega^3-2\omega)}{(\omega^4+\omega^2)(4+\omega^2)}$$
$$=\frac{-30\omega^2}{(\omega^4+\omega^2)(4+\omega^2)}+j\frac{10(\omega^3-2\omega)}{(\omega^4+\omega^2)(4+\omega^2)}$$

Equating respectively the real and imaginary parts,

$-\frac{-30\omega^2}{(\omega^4+\omega^2)(4+\omega^2)}=0;$ from which, $\omega=\infty$

and $\frac{10(\omega^3-2\omega)}{(\omega^4+\omega^2)(4+\omega^2)}=0;$ from which $\omega^3-2\omega=0$ or $\omega=\pm\sqrt{2}$

Magnitude of $G(j\omega)=|G(j\omega)|=\frac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}}$

Phase angle of $G(j\omega)=|G(j\omega)|=\left(-90^\circ-\tan^{-1}\omega-\tan^{-1}\frac{\omega}{2}\right)$

$G(j\omega)$ acts the real axis at $\omega=\pm\sqrt{2}$

The value of $G(j\omega)$ at $\omega=\pm\sqrt{2}$ is calculated as

Problems

$$-\frac{-30\omega^2}{(\omega^4+\omega^2)(4+\omega^2)} = -\frac{30\times 2}{(4+2)(4+2)} = -\frac{60}{36} = -1.67$$

The values of $G(j\omega)$, i.e. its magnitude and phase at $\omega \to 0$ and $\omega \to \infty$ are calculated as

$$\lim_{\omega\to 0}|G(j\omega)| = \lim_{\omega\to 0}\frac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}} = \infty$$

$$\lim_{\omega\to\infty}|G(j\omega)| = \lim_{\omega\to\infty}\frac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}} = 0$$

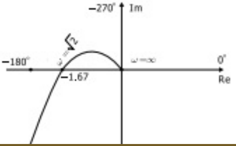
$$\lim_{\omega\to 0}\angle G(j\omega) = \lim_{s\to 0}\left(-90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}\right) = -90^\circ - 0^\circ - 0^\circ = -90^\circ$$

$$\lim_{\omega\to\sqrt{2}}\angle G(j\omega) = \lim_{\omega\to\sqrt{2}}\left(-90^\circ - \tan^{-1}\sqrt{2} - \tan^{-1}\frac{1}{\sqrt{2}}\right) = -180^\circ$$

$$\lim_{\omega\to\infty}\angle G(j\omega) = \lim_{\omega\to\infty}\left(-90^\circ - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}\right) = -90^\circ - 90^\circ - 90^\circ = -270^\circ$$

Thus the informations for drawing the polar plot are shown below. The polar plot is shown in Fig.

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0	∞	-90°
$\sqrt{2}$	-1.67	-180°
∞	0	-270°



Sketch the Bode plot and hence determine the gain cross over frequency and phase cross over frequency for the transfer function of a system represented by

$$G(s)=\frac{10}{s(s+0.5s)(1+0.1s)}$$

Solution

By putting $s=j\omega$ in the transfer function we have

$$G(j\omega)=\frac{10}{j\omega(1+j0.5\omega)(1+j0.1\omega)}$$

a) The corner frequencies are:

$$\omega_1=\frac{1}{0.5}$$
$$=2 \text{ rad/sec}$$
$$\omega_2=\frac{1}{0.1}$$
$$=10 \text{ rad/sec}$$

b) As per procedure, the starting frequency of Bode plot is taken as lower than the lowest corner frequency. Here the lowest corner frequency is 2 rad/sec. Therefore, we can choose starting frequency as 1 rad/sec.

c) The system is type 1 system (type of the system is indicated by the power of s in the denominator of the transfer function). So the initial slope of the Bode plot is −20 dB/decade. This initial slope will continue till the lowest corner frequency of $\omega=2$ rad/sec.

d) The corner frequency of $\omega=2$ rad/sec is due to the term $\frac{1}{1+j0.5\omega}$ of the transfer function. Therefore, the slope of the Bode plot will change by another −20 dB/decade after the corner frequency of $\omega=2$ rad/sec. The total slope, therefore, will become −40 dB/dec. This slope will continue till the next corner frequency of $\omega=10$ rad/sec. The corner frequency of $\omega=10$ rad/sec is due to the term $\frac{1}{1+j0.1\omega}$ of the transfer function. The slope of the Bode plot after the corner frequency of 10 rad/sec will change by another −20 dB/dec. Therefore, after corner frequency of 10 rad/sec, the total slope of the Bode plot becomes −60 dB/dec. and this slope will continue for higher frequencies.

e) The magnitudes of the Bode plot at different frequencies are calculated as shown below.

Problems

Frequency in rad/sec	Magnitude in dB
$\omega = 1$	$\left\lvert\frac{K}{j\omega}\right\rvert = 20\log K - 20\log\omega$ $= 20\log 10 - 20\log 1$ $= 20$ dB Here $K = 10$, given
$\omega = 2$	$\left\lvert\frac{K}{j\omega(1+j\omega 0.5)}\right\rvert = 20\log 10 - 20\log\omega$ $- 20\log\sqrt{1^2+2.5\omega^2}$ $= 20 - 20\log 2 - 10\log 2$ $= 11$ dB
$\omega = 10$	$\left\lvert\frac{K}{j\omega(1+j0.5\omega)(1+j0.1\omega)}\right\rvert = 20\log K - 20\log\omega$ $- 20\log\sqrt{1^2+.25\omega^2}$ $- 20\log\sqrt{1^2+0.1\omega^2}$ $= -16.16$ dB

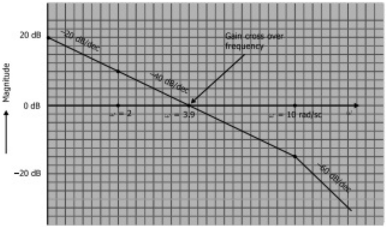


Fig. *Bode magnitude plot for* $G(s) = \frac{10}{s(s+j0.5s)(1+0.1s)}$

f) Now, we will calculate the angle $\angle G(j\omega)$ for frequencies of $\omega = 1$ rad/sec to $\omega = 15$ rad/sec (say). The calculations are done using the relation, $\angle G(j\omega) = -90° - \tan^{-1} 0.5\,\omega - \tan^{-1} 0.1\,\omega$ (from the system transfer function)

ω	0	0.1	1.0	2	5	10	15
ϕ	−90°	−93.43°	−122.3°	−146.31°	−184.76°	−213.7°	−228.7°

g) We now sketch the Bode plot for magnitude and phase angle as shown in Figs. 10.17 and 10.18. From the figure we find,
Gain cross over frequency = 3.9 rad/sec
Phase cross over frequency = 4.5 rad/sec

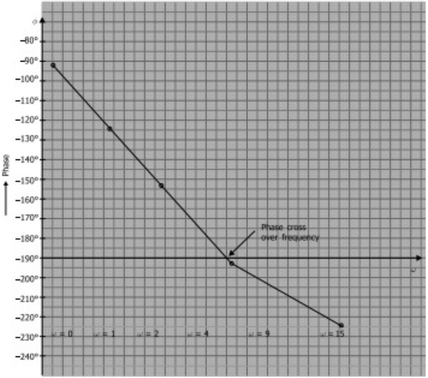


Fig. *Bode phase plot for* $G(s) = \dfrac{10}{s(s+j0.5s)(1+0.1s)}$

Example 10.4 Find the open-loop transfer function of a system whose approximate Bode plot is shown below.

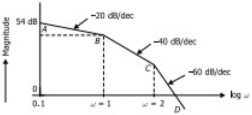


Fig. 10.19

Solution

a) The initial slope of the Bode plot is −20 dB/dec. We can write the equation of the line AB in the form $y = mx + c$*. Here, we write

$$y = -20 \log \omega + c$$

We put $\omega = 0.1$ and $y = 54$ dB and calculate the value of c as

$$54 \text{ dB} = -20 \log (0.1) + c$$

or, $$c = (54 - 20) \text{ dB} = 34 \text{ dB.}$$

For a slope of −20 dB/decade, the system is type 1 and its transfer function is $\frac{K}{j\omega}$. We have to find the value of K

$$c = 20 \log K$$

or, $$34 = 20 \log K$$

or, $$\log K = 1.7$$

$$K = 50.12$$

Therefore for the initial Bode plot we get the transfer function,

$$\frac{K}{s} = \frac{\text{[illegible]}\ .2}{s}$$

[*We know that starting part of Bode plot gives the transfer function of $\frac{K}{j\omega}$. This can be expressed in dB form, $T(s) = \frac{K}{s}$ or $T(j\omega) = \frac{K}{j\omega}$. The magnitude representation in dB form is

$$|T(j\omega)| = y = 20 \log K - 20 \log \omega.$$

Problems

or, $y = -20\log\omega + 20\log K.$

Viewing in $y = -mx + c$ form, $c = 20\log K]$

At corner frequency, $\omega = 1$ the slope has changed by another -20 dB/dec. The slope is negative. The corresponding factor of the T.F is $\frac{1}{(1+s)}$.

At the corner frequency $\omega = 2$, the slope is increased by another -20 dB/dec. The slope is negative. Hence the corresponding factor of the T.F. is $\frac{1}{(1+0.5s)}$.

Thus the transfer function of the control system is

$$G(s) = \frac{K}{s(1+s)(1+0.5s)} = \frac{50.2}{s(1+s)(1+0.5s)}$$

Problems

Example 10.10 (a) State the advantages of Bode plot. Draw the Bode diagram for

$$G(s) = \frac{100(0.02s+1)}{(s+1)(1+0.1s)(1+0.01s)^2}$$

(b) Mark the following on the Bode diagram, recording the numerical values

a) Gain cross over frequency
b) Phase margin
c) Phase cross over frequency
d) Gain margin

Solution

(a) In Bode plot, $G(j\omega)$ in dB, i.e. $20\log|G(j\omega)|$ is plotted against log ω. Similarly, phase angle of $G(j\omega)$ is plotted against log ω. The following are its advantages:

i) Since $G(j\omega)$ consists of many multiplicative factors in both numerator and denominator, it is convenient to take logarithm of $|G(j\omega)|$ so that the factors can be converted into additions and subtractions, which can be carried out easily. This helps in simplified design modification procedure.
ii) The relative stability of the system can be studied by calculating gain margin and phase margin from the Bode plot.
iii) Transfer function can be obtained from Bode plot.
iv) The value of system gain K can be determined from Bode plot for desired gain margin and phase margin specifications.

Solution

(b) The students are to try this problem following the steps as:

First the corner frequencies are calculated and the initial slope is calculated as

$$20\log_{10} K = 20\log_{10} 100 = 40 \text{ dB}.$$

Step 1: Arrange $G(s)\,H(s)$ in time constant form.

The given system is already in time constant form, i.e.

$$G(s) = \frac{100(0.02s+1)}{(s+1)(1+0.1s)(1+0.01s)^2} \quad \text{....(1)}$$

Step 2: Factors

As the system is type zero system $20\log_{10}|G(j\omega)H(j\omega)|$

$$20\log_{10} K = 20\log_{10}(\omega)$$
$$= 20\log_{10} = 40 \text{ dB}$$

The following are the corner frequencies:

i) Simple zero $(0.02s + 1)$, $\omega_{c_1} = \frac{1}{0.02} = 50$ rad/sec

ii) Simple pole $(s + 1)$, $\omega_{c_2} = 1$ rad/sec

iii) Simple pole $(1 + 0.1s)$, $\omega_{c_3} = 10$ rad/sec

iv) Simple pole $(1 + 0.01s)^2$, $\omega_{c_4} = 100$ rad/sec

Step 3: Magnitude plot

i) Contribution of K is $20\log K = 40$ dB

ii) 1 pole at origin $(s + 1)$ starting slope becomes $= -20$ dB/decade

iii) At $\omega_{c_2} = 10$

Slope will be $= -40$ dB/decade

iv) At $\omega_{c_1} = 50$

Slope will be $-40 + 20 = -20$ dB/decade

v) At $\omega_{c_4} = 100$

Slope will be $= -20 - 40 = -60$ dB/decade.

Step 4: Phase angle plot

Substitute $s = j\omega$ in equation (1)

$$G(j\omega)H(j\omega) = \frac{100(0.02j\omega+1)}{(j\omega+1)(1+0.1j\omega)(1+0.01j\omega)}$$

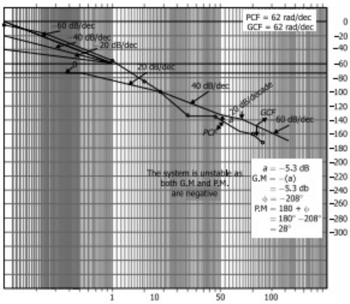
$$\angle G(j\omega)H(j\omega) = \frac{\angle 1+0.02j\omega}{\angle + j\omega\angle 1+0.1j\omega < 1+0.01j\omega}$$

The calculation of phase angle is shown in a tabular form. The Bode plot for the given *TF* is shown in Fig. 10.26.

Problems

Phase angle table

ω	$\tan^{-1} 0.02\omega$	$-\tan^{-1} \omega$	$-\tan^{-1} 0.1\omega$	$-2\tan^{-1} 0.01\omega$	ϕ_R
0.1	0.114	−5.71	−0.57	−0.114	−6.28
1	1.145	−45	−57	−1.14	−50.695
5	5.71	−78.69	−26.56	−5.72	−105.26
10	11.30	−84.28	−45	−11.4	−129.38
20	21.80	−87.13	−63.43	−45.22	−173.98
50	45	−88.85	−78.69	−53.12	−175.16
100	63.43	−89.42	−84.28	−90	−200.02
150	71.56	−89.42	−86.42	−112.61	−216.84
∞	90°	−90°	−90°	−180°	−360°



Concept of Controllers

Introduction

- In control systems, a controller is a mechanism that seeks to minimize the difference between the actual value of a system (i.e. the process variable) and the desired value of the system (i.e. the setpoint). Controllers are a fundamental part of control engineering and used in all complex control systems.

Important uses of the Controllers

- Controllers improve the steady-state accuracy by decreasing the steady state error.
- As the steady-state accuracy improves, the stability also improves.
- Controllers also help in reducing the unwanted offsets produced by the system.
- Controllers can control the maximum overshoot of the system.
- Controllers can help in reducing the noise signals produced by the system.
- Controllers can help to speed up the slow response of an overdamped system.

Types of Controllers

In the continuous controller theory, there are three basic modes on which the whole control action takes place, which are:

- Proportional controllers.
- Integral controllers.
- Derivative controllers.

These three types of controllers can be combined into new controllers:

- Proportional and integral controllers (PI Controller)
- Proportional and derivative controllers (PD Controller)
- Proportional integral derivative control (PID Controller)

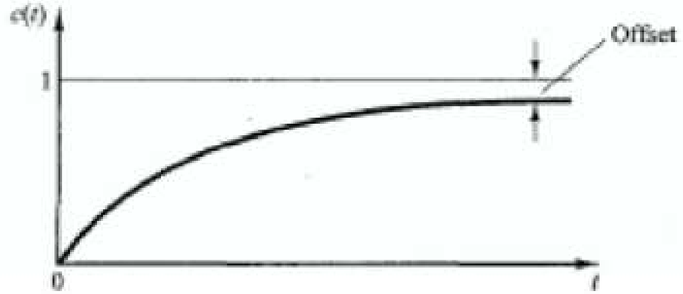
Proportional controllers

To discuss proportional controllers, as the name suggests in a proportional controller the output (also called the actuating signal) is directly proportional to the error signal. Now let us analyze the proportional controller mathematically. As we know in proportional controller output is directly proportional to the error signal, writing this mathematically we have, here proportionality constant is called as proportional controller Kp

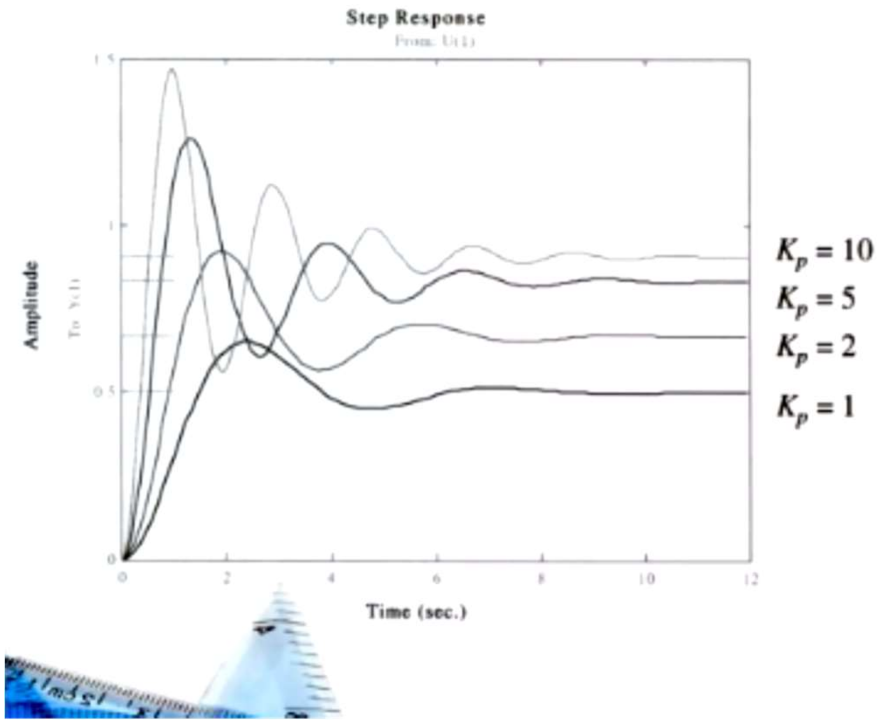
$$A(t) \propto e(t)$$

Proportional controllers

- Actuating signal is proportional to error signal
- Such a system always has a steady-state error in the step response. Such a steady-state error is called an offset.



Change in gain in P controller



- **Increase in gain**:

  → Upgrade both steady-state and transient responses

  →Increases oscillations

  → Reduce steady-state error

  → ***Reduce stability!***

Advantages of Proportional Controller

- Now let us discuss some advantages of the proportional controller.
- The proportional controller helps in reducing the steady-state error, thus makes the system more stable.
- The slow response of the overdamped system can be made faster with the help of these controllers.

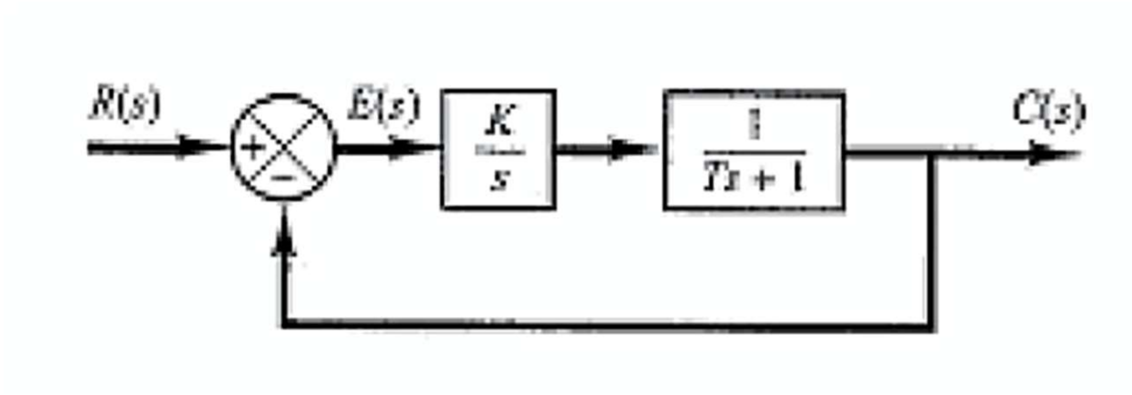
Disadvantages of Proportional Controller

Now there are some serious disadvantages of these controllers and these are written as follows:

1. Due to the presence of these controllers, we get some offsets in the system.
2. Proportional controllers also increase the maximum overshoot of the system.

Integral Controllers

- As the name suggests in integral controllers the output (also called the actuating signal) is directly proportional to the integral of the error signal.



- Where K/s (Ki) is an integral constant also known as controller gain. The integral controller is also known as reset controller.

Integral Controller

- Integral of error with a constant gain
  - → increase the system type by 1
  - →*eliminate steady-state error for a unit step input*
  - → amplify overshoot and oscillations

Advantages of Integral Controller

- Due to their unique ability, Integral Controllers can return the controlled variable back to the exact set point following a disturbance that's why these are known as reset controllers.

Disadvantages of Integral Controller

- It tends to make the system unstable because it responds slowly towards the produced error.

Derivative Controllers

- It should be used in combinations with other modes of controllers because of its few disadvantages which are written below:
- It never improves the steady-state error.
- It produces saturation effects and also amplifies the noise signals produced in the system.
- Now, as the name suggests in a derivative controller the output (also called the actuating signal) is directly proportional to the derivative of the error signal.

Derivative Controller

- Differentiation of error with a constant gain

  $\rightarrow$ detect rapid change in output

  $\rightarrow$ *reduce overshoot and oscillation*

$\rightarrow$ do not affect the steady-state response

Advantages of Derivative Controller

- The major advantage of a derivative controller is that it improves the transient response of the system.

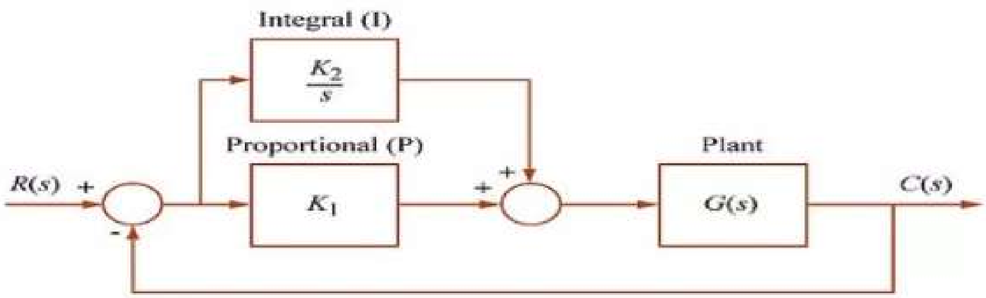
Proportional and Integral Controller

- As the name suggests it is a combination of proportional and an integral controller the output (also called the actuating signal) is equal to the summation of proportional and integral of the error signal.
- Through the PI controller, we are adding one pole at origin and one zero somewhere away from the origin (in the left-hand side of complex plane).
- As the pole is at the origin, its effect will be more, hence PI controller may reduce the stability; but its main advantage is that it reduces steady-state error drastically, due for this reason it is one of the most widely used controllers.

Proportional-Plus-Integral Control

- **To** eliminate offset, the proportional controller may be replaced by a proportional-plus-integral controller.

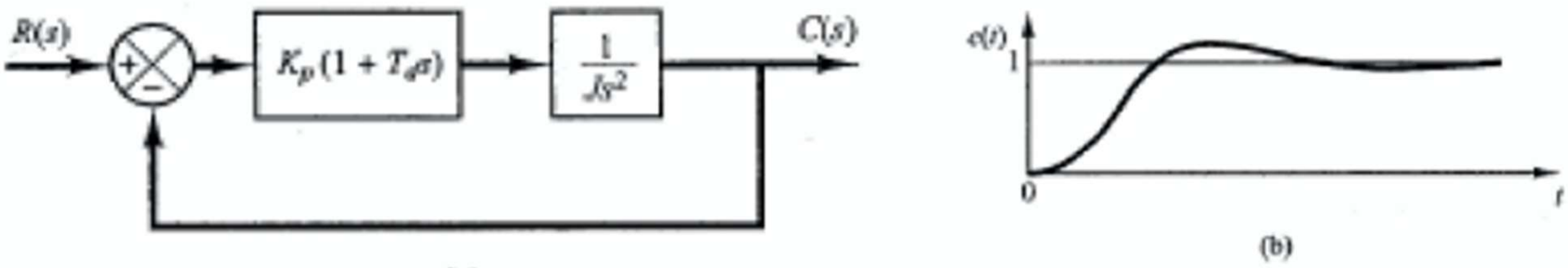
- If integral control action is added to the controller, then, as long as there is an error signal, a signal is developed by the controller to reduce this error, provided the control system is a stable one.



Proportional and Derivative Controller

- As the name suggests it is a combination of proportional and a derivative controller the output (also called the actuating signal) is equals to the summation of proportional and derivative of the error signal.
- Generally, it is said, PD controller improves transient performance and the PI controller improves the steady-state performance of a control system.

Proportional-Plus-Derivative Control



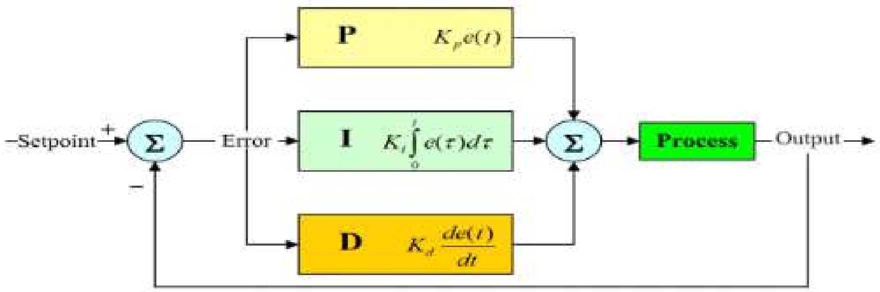
(a) (b)

$$\frac{C(s)}{R(s)} = \frac{K_p(1 + T_d s)}{Js^2 + K_p T_d s + K_p}$$

Proportional-Plus-Derivative -Plus-Integral Control

- A proportional–integral–derivative controller (PID controller) is a control loop feedback mechanism widely used in industrial control systems – a PID is the most commonly used feedback controller.
- A PID controller calculates an "error" value as the difference between a measured process variable and a desired setpoint. The controller attempts to minimize the error by adjusting the process control inputs.

- The PID controller calculation involves three separate constant parameters, and is accordingly sometimes called three-term control: the proportional, the integral and derivative values, denoted P, I, and D.
- These values can be interpreted in terms of time: P depends on the present error, I on the accumulation of past errors, and D is a prediction of future errors, based on current rate of change.



Parameter	P	PI	PD	PID
Time constant	Decrease	Increase	Decrease	Small Change(Decrease)
Rise Time	Decrease	Increase	Decrease	Small change(Decrease)
Peak Time	Decrease	Increase	Decrease	Small change(Decrease)
Overshoot	Increase	Decrease	Increase	Small change(Decrease)
Settling Time	Small Change	Increase	Decrease	Small change(Decrease
Steady State Error	Decrease	Decrease	No Change	Decrease
Stability	Decrease(worse)	Decrease(worse)	Improve	Improve

Comparison between P PI and PID controller.

Control Systems

Unit IV: Compensators

Sub-code: 18EE502
Department of Electrical Engineering
Bapatla Engineering College
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Compensator

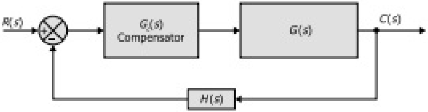
Compensator is an additional component or circuit that is inserted into a control system to equalize or compensate for a deficient performance.
Necessities of compensation
1. In order to obtain the desired performance of the system, we use compensating networks. Compensating networks are applied to the system in the form of feed forward path gain adjustment.
2. Compensate a unstable system to make it stable.
3. A compensating network is used to minimize overshoot.
4. These compensating networks increase the steady state accuracy of the system. An important point to be noted here is that the increase in the steady state accuracy brings instability to the system.
5. Compensating networks also introduces poles and zeros in the system thereby causes changes in the transfer function of the system. Due to this, performance specifications of the system change.

Types of Compensator

Series or Cascade compensation
Parallel or feedback compensation
Combined Cascade and feedback compensation or Series parallel compensator

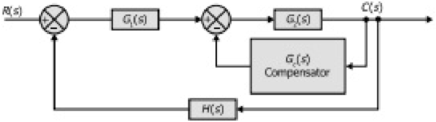
Series or Cascade compensation

Compensator can be inserted in the forward path as shown in fig below. The transfer function of compensator is denoted as $G_c(s)$, whereas that of the original process of the plant is denoted by G(s).



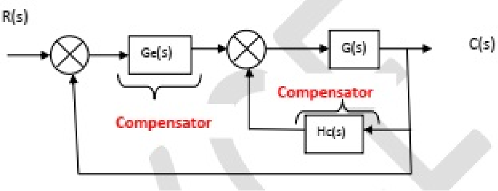
Parallel or feedback compensation

The feedback is taken from some internal element and compensator is introduced in such a feedback path to provide an additional internal feedback loop. Such compensation is called feedback compensation or parallel compensation. The arrangement is shown in fig.



Combined Cascade and feedback compensation or Series parallel compensator

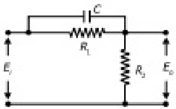
In some cases, it is necessary to provide both types of compensations, series as well as feedback. Such a scheme is called series – parallel compensation. The arrangement is shown in fig. below.



Compensating Network

A compensating network is one which makes some adjustments in order to make up for deficiencies in the system. Compensating devices are may be in the form of electrical, mechanical, hydraulic etc. Most electrical compensator is RC filter. The simplest networks used for electrical compensator are Lead compensator – (to speed up transient response, margin of stability and improve error constant in a limited way) Lag compensator – (to improve error constant or steady-state behavior – while retaining transient response) Lead – Lag compensator – (A combination of the above two i.e. to improve steady state as well as transient)

A system which has one pole and one dominating zero (the zero which is closer to the origin than all orver zeros is known as dominating zero.) is known as lead network. The basic requirement of the phase lead network is that all poles and zeros of the transfer function of the network mast lie on (-)ve real axis interlacing each other with a zero located at the origin of nearest origin. Given below is the circuit diagram for the phase lead compensation network.



Phase Lead Compensation Network Transfer Function

$$= G_c(s) = \frac{e_0(s)}{e_c(s)} \; \frac{e_0(s)}{e_i(s)} = \frac{R_2}{R_2 + \frac{R_1 \times \frac{1}{Cs}}{R_1 + \frac{1}{Cs}}} \begin{array}{l} = \dfrac{R_2}{R_2 + \frac{R_1}{R_1 cs + 1}} \\ = \dfrac{R_2\left(R_1 Cs + 1\right)}{R_1 + R_2\left(R_1 Cs + 1\right)} \end{array} \; \frac{e_0(s)}{e_i(s)} = \frac{R_2(R_1Cs}{R_1 + R_2(R_1C}$$

$$\frac{e_0(s)}{e_i(s)} = \frac{R_2(R_1Cs+1)}{R_1R_2Cs+R_1+R_2} = \frac{R_2}{R_1+R_2}\left[\frac{R_1Cs+1}{1+\frac{R_2}{R_1+R_2}R_1Cs}\right] = \alpha\left[\frac{1+Ts}{1+\alpha Ts}\right] - - - - -$$

$- - - - -(1)$ Where $\alpha = \frac{R_2}{n_1+\pi_2} < 1$ $T = R_1C$

Phase Lead Compensation

Equation 1 can be written in the form of

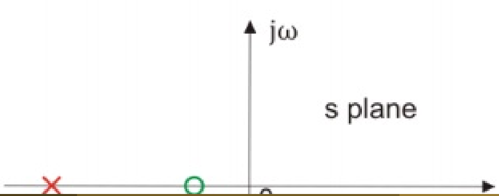
$$G_c(s) = \frac{\alpha T\left(s+\frac{1}{T}\right)}{\alpha T\left(s+\frac{1}{\alpha T}\right)}$$

$$G_c(s) = \frac{\left(s+\frac{1}{T}\right)}{s+\frac{1}{\alpha T})}$$

$$= \frac{s+Z_c}{S+P_c}$$

$$\text{Where } Z_c = \frac{1}{T} \text{ and } P_c = \frac{1}{\alpha T}$$

Let us draw the pole zero plot for the above transfer function.



The sinusoidal transfer function of the lead network is obtained by substituting S − jω in equation 1 $G_c(j\omega) = \frac{e_0(j\omega)}{e_i(j\omega)} = \frac{\alpha(1+j\omega T)}{(1+j\omega\omega T)}$ Let $|G_c(j\omega) = \emptyset =| \frac{\omega_0(j\omega)}{e_c(j\omega)} \emptyset = \lfloor G_c(j\omega) = \tan^{-1}\omega T - \tan^{-1}\omega_\omega \alpha T - - - - 2$ as $\alpha < 1$ we have,

$$\tan^{-1}\omega\alpha T < \tan^{-1}\omega T$$

Φ is always positive Therefore the output voltage always lead the input voltage in above network. Hence the above network is called lead network Effect of Phase Lead Compensation

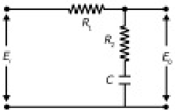
1. The velocity constant $\mathrm{K_s}$ increases.
2. The slope of the magnitude plot reduces at the gain crossover frequency so that relative stability improves and ercor decrease due to error is directly proportional to the slope.
3. Phase margin increases.
4. Response becomes faster.

Phase Lead Compensation

Advantages of Phase Lead Compensation
1. Due to the presence of phase lead network the speed of the system increases because it shifts gain crossover frequency to a higher value.
2. Due to the presence of phase lead compensation maximum overshoot of the system decreases.
Disadvantages of Phase Lead Compensation
1. Steady state error is not improved.

Phase Lag Compensation

A system which has one zero and one dominating pole (the pole which is closer to origin that all other poles is known as dominating pole) is known as lag network. The basic requirement of the phase lag network is that all poles and zeros of the transfer function of the network must lie in (-)ve real axis interlacing each other with a pole located or on the nearest to the origin. Given below is the circuit diagram for the phase lag compensation network.



Transfer Function

$$\text{Transfer Function } = G_c(s) = \frac{e_0(s)}{e_i(s)}$$

$$\frac{e_0(s)}{e_i(s)} = \frac{\left[R_2+\frac{1}{Cs}\right]l_{(s)}}{\left[R_1+R_2+\frac{1}{Cs}\right]I_{(s)}}$$

$$= \frac{R_2Cs+1}{1+(R_1+R_2)Cs}$$

$$\frac{R_2Cs+1}{1+\left(\frac{R_1+R_2}{R_2}\right)R_2Cs}$$

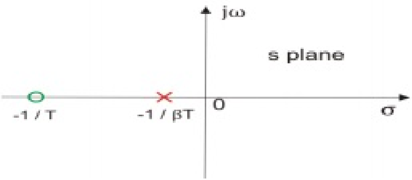
$$\text{Let } \tau - R_2C \text{ and } \beta = \frac{R_1+R_2}{R_2} > 1$$

$$G_c(s) = \frac{1+st}{1+\beta st} - - - - - - - 1$$

$$G_c(s) = \frac{e_0(s)}{e_i(s)} = \frac{1}{\beta}\left[\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}\right] = \frac{1}{\beta}\left[\frac{s+Z_c}{s+P_c}\right]$$

$$\text{Where } Z_c = \frac{1}{\tau} \text{ and } P_c = \frac{1}{t\beta}$$

The pole zero location of the lag network is as shown in figure below.



To obtain sinusoidal transfer function we put s-jw in the equation 1

$$G_c(j\omega) = \frac{e_0(j\omega)}{e_l(j\omega)} = \frac{1 + j\omega t}{1 + j\omega\beta t}$$

If $\emptyset_{\mathrm{m-1}\sigma(j\omega)}$ then

$$\emptyset = \lfloor G_c(j\omega) = \tan^{-1}\omega\tau - \tan^{-1}\omega\beta t - - - -2$$

Since $\beta > 1, \tan^{-1}\omega\beta t > \tan^{-1}\omega t$ Or ϕ_a is negative Therefore the output voltage lags the input voltage. Hence the name lag Network.

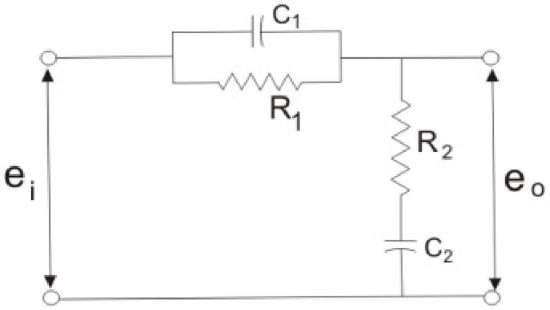
Effect of Phase Lag Compensation
1. Gain crossover frequency increases.
2. Bandwidth decreases.
3. Phase margin will be increase.
4. Response will be slower before due to decreasing bandwidth, the rise time and the settling time become larger.

Advantages of Phase Lag Compensation
Let us discuss some of the advantages of phase lag compensation -
1. Phase lag network allows low frequencies and high frequencies are attenuated.
2. Due to the presence of phase lag compensation the steady state accuracy increases.

Disadvantages of Phase Lag Compensation
Some of the disadvantages of the phase lag compensation -
1. Due to the presence of phase lag compensation the speed of the system decreases.

Phase Lag Lead Compensation

With single lag or lead compensation may not satisfied design specifications. For an unstable uncompensated system, lead compensation provides fast response but does not provide enough phase margin whereas lag compensation stabilize the system but does not provide enough bandwidth. So we need multiple compensators in cascade. Given below is the circuit diagram for the phase lag- lead compensation network.



Now let us determine transfer function for the given network and the transfer function can be determined by finding the ratio of the output voltage to the imput voltage.

$$G_c(s) = \frac{\left(s + \frac{1}{\tau_1}\right)}{\left(s + \frac{1}{\beta\tau_1}\right)\left(s + \frac{1}{\alpha\tau_2}\right)} \quad \alpha < 1, \beta > 1$$

$$G_c(s) = \frac{(1 + s\tau_1)(1 + s\tau_2)/\tau_1\tau_2)}{s^2 + s\left(\frac{1}{\beta\tau_1} + \frac{1}{\alpha\tau_2}\right) + \frac{1}{\alpha\beta\tau_1\tau_2}}$$

$$= \frac{(1 + s\tau_1)(1 + s\tau_2)}{\tau_1\tau_2 s^2 + s\left(\frac{\tau_1}{\alpha} + \frac{\tau_2}{\beta}\right) + \frac{1}{\alpha\beta}} - - - - - - - - - 1$$

Phase Lag Lead Compensation

We have,

$$e_0(s) = \left[R_2 + \frac{1}{C_2 s}\right] I(s)$$

$$e_i(s) = \left[\frac{R_1 \frac{1}{C_1 s}}{R_1 + \frac{1}{C_1 s}} + R_2 + \frac{1}{C_2 s}\right] I(s)$$

$$\frac{e_i(s)}{\beta e_o(s)} = \frac{R_1}{R_1 C_1 s + 1} + R_2 + \frac{1}{C_2 s}\Bigg]$$

$$\frac{e_1(s)}{e_e(s)} = \frac{R_1 C_1 s + (R_2 C_2 s + 1)(R_1 C_1 s + 1)}{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}$$

$$\Bigg(G_c(s) = \frac{R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_1 R_2 C_1 C_2 s^2 + s(R_1 C_1 + R_2 C_2 + R_1 C_2) + 1} - - - -2$$

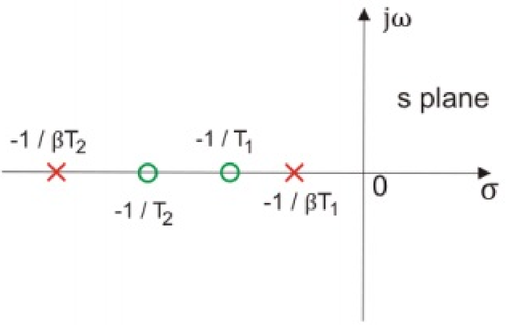
Comparing equation 1 and 2

$$\tau_1 = R_1C_1 \text{ and } \tau_2 = R_2C_2$$
$$\frac{\tau_1}{\alpha} + \frac{\tau_2}{\beta} = R_1C_1 + R_2C_2 + R_1C_2$$
$$\frac{1}{\alpha\beta} = 1 \text{ therefore } \alpha\beta = 1$$

A single lag- lead network doesnot permit an independent choice of α and β

$$G_c(s) = \frac{\left(s + \frac{1}{\tau_1}\right)}{\left(s + \frac{1}{\beta\tau_1}\right)} - \frac{\left(s + \frac{1}{\tau_2}\right)}{\left(s + \frac{1}{\beta\tau_2}\right)} \text{ means } \frac{1}{\alpha} = \beta \text{ and } \beta > 1$$

The pole zero location of the lag network is as shown in figure below.



Advantages of Phase Lag Lead Compensation

Let us discuss some of the advantages of phase lag- lead compensation-

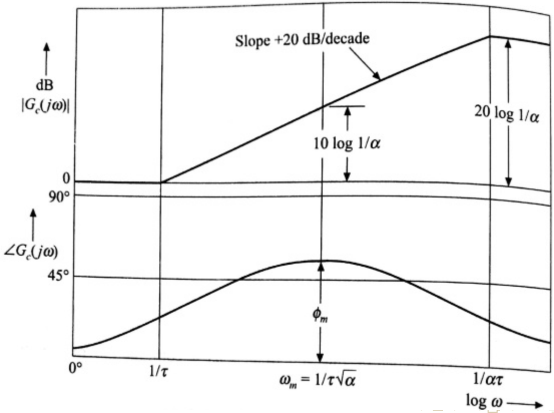
1. Due to the presence of phase lag-lead network the speed of the system increases because it shifts gain crossover frequency to a higher value.
2. Due to the presence of phase lag-lead network accuracy is improved.

Bode plot of Lead Compensation

The Bode diagram of the lead compensator is shown in figure. The phase-lead of the above compensator at any frequency ω is given by

$$\phi = \tan^{-1}\omega\tau - \tan^{-1}\alpha\omega\tau$$

$\tan\phi = \frac{\omega\tau - \alpha\omega\tau}{1+\omega\tau\cdot\alpha\omega\tau} = \frac{\omega\tau(1-\alpha)}{1+\alpha\omega^2\tau^2}$



Bode plot of Lead Compensation

The frequency at which maximum phase-lead occurs ω_m is given by the solution of $\frac{d\phi}{d\omega} = 0$, i.e. i.e.

$$\frac{d}{d\omega}\left(\frac{\omega\tau(1-\alpha)}{1+\alpha\omega^2\tau^2}\right) = 0$$

or

$$\left(1+\alpha\omega^2\tau^2\right)(1-\alpha)\tau - \omega\tau(1-\alpha)\left(2\alpha\omega\tau^2\right) = 0$$

i.e.

$$\omega_m = \frac{1}{\tau\sqrt{\alpha}} = \sqrt{(1/\tau)(1/\alpha\tau)}$$

So, ω_m is the geometric mean of the two corner frequencies of the compensator. At $\omega = \omega_m$, the maximum phase-lead ϕ_m is given by

$$\tan\phi_m = \frac{\omega_m\tau(1-\alpha)}{1+\alpha\omega_m^2\tau^2} = \frac{\frac{1}{\tau\sqrt{\alpha}}\cdot\tau\cdot(1-\alpha)}{1+\alpha\cdot\frac{1}{\tau^2\alpha}\cdot\tau^2} = (1-\alpha)/2\sqrt{\alpha}$$

or

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

Compensation design can be carried out in time-domain or frequency-domain. The specifications in time-domain are generally given in the following form:

1. Damping ratio ξ
2. Peak overshoot M_p-indicative of the relative stability
3. Undamped natural frequency ω_n
4. Rise time t_r or settling time t_s-indicative of speed of response
5. Error constant e_{ss}-indicative of steady-state error

Frequency-domain specifications are generally given in the following form.

1. Phase margin ϕ_{pm} or resonant peak M_r-indicative of relative stability
2. Bandwidth ω_b or resonant frequency ω_r-indicative of rise time and settling time
3. Error constant indicative of steady-state error

Compensation design

The design in time-domain is carried out using the root locus and the design in frequency domain is carried out using the Nyquist plots, Bode plots, or Nichols chart. Design in time is tedious. So, usually the given time-domain specifications are usually first converted into frequency-domain specifications using second-order correlations between time and frequency responses, the design and compensation is carried out in frequenc domain and then the results are back converted into time domain. The correlations discussed earlier are as follows:

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

$$\phi_{pm} = \tan^{-1}\left\{2\xi/\left[\sqrt{(1+4\xi^4)}-2\xi^2\right]^{1/2}\right\}$$

$$\omega_b = \omega_n\left[1-2\xi^2+\sqrt{(2-4\xi^2+4\xi^4)}\right]^{1/2}$$

Compensation design

The specifications in terms of M_r and ω_r are convenient for compensation using the Nyquist plots. When phase margin is specified, the Bode plots are more convenient. Gain crossover frequency ω_g can be used as a rough measure of bandwidth ω_b. When the Nichols charts are used any type of specification can be handled.

The advantages of frequency-domain compensation are as follows:

1. Simplicity in analysis and design
2. Ease in experimental determination of frequency response for real systems

The disadvantage of frequency-domain compensation is that direct control on system performance is lost.

Even though frequency-domain compensation can be carried out using the Nyquist plots, Bode plots or Nichols chart, the compensation is normally carried out by using the Bode plots because of the following:

1. The Bode plots are easier to draw and modify.
2. The gain adjustment can be conveniently carried out using the Bode plots.

The lead compensator is basically a high-pass filter. The design procedure for a lead compensator given below is quite general and applies to any type and order of a system. Step 1. Adjust the system error constant to the desired value. Determine the open-loop gain K required to satisfy the specified error constant.
Step 2. Using this value of K draw the magnitude and phase Bode plots and determine the phase margin ϕ_{pm1} and gain crossover frequency ω_{g1} of the uncompensated system. If the phase margin of the uncompensated system ϕ_{pm1} is not satisfactory, proceed with the following steps to design a lead compensator.
Step 3. Determine the phase-lead $\phi_{/}$required using the relation

$$\phi_l = \phi_s - \phi_{pm1} + \epsilon$$

where ϕ_s is the required (specified) phase margin, ϕ_{pm1} phase margin of the fixed part of the system (i.e. the uncompensated system); and ϵ is margin of safety.

step 4. Let $\phi_m = \phi_l$ and determine the α parameter of the network using formula

$$\alpha = \frac{1 - \sin \phi_m}{1 + \sin \phi_m}$$

If the required ϕ_m is more than 60°, it is recommended to use two identical networks each contributing a maximum lead of $\phi_V/2$. (It is because in order to provide a phase-lead of ϕ_1 at the new gain crossover frequency ω_{g2} with the largest value of α, the frequency of maximum phase-lead ω_m of the network must be made to coincide with ω_{g2}. Thus we get $\omega_{g2} = \omega_m$)

Step 5. Calculate the dB-gain $10 \log(1/\alpha)$. Locate the frequency at which the uncompensated system has a gain of $-10 \log(1/\alpha)$. This is the frequency $\omega_{g2} = \omega_m$ of the compensated system.

Steps for Lead Compensation design

step 6. Compute the two corner frequencies of the network as

$$\omega_1 = 1/\tau = \omega_m\sqrt{\alpha}; \omega_2 = 1/\alpha\tau = \omega_m/\sqrt{\alpha}$$

With those values of ω_1 and ω_2, the design is complete and the lead compensator transfer function can be written as

$$G_c(s) = \frac{1+\tau s}{1+\alpha\tau s}$$

step7. Draw the magnitude and phase bodeplot of the compensated system and check the resulting phase margin and gain crossover frequency (it is a rough measure of bandwidth of the system), if the phase margin is still low raise the value of ϵ and repeat from step 3 above.
step8. Check any addition specifications on the system performance eg. bandwidth, redesign for another choice of crossover frequency till specifications meet.

STATE SPACE ANALYSIS

Introduction

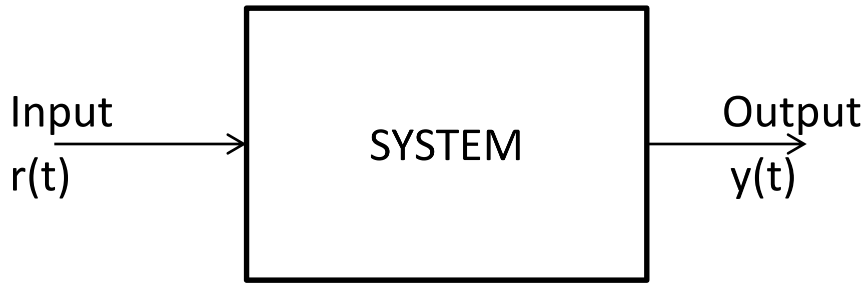
- State space analysis is an powerful and modern approach for the design and analysis of control systems.
- The conventional or old methods for the design and analysis of control systems is based on transfer function method.
- The transfer function method for design and analysis had many drawbacks such as..
  - Transfer function is defined under zero initial conditions
  - Applicable to LTI systems
  - SISO systems
  - Does not provide the information regarding internal state of the system
- Initial conditions can be incorporated in the system design
- State equations are highly compatible for simulation on analog or digital computers

Advantages of state variable analysis.

This can be applicable to

- Linear systems
- Non-linear system
- Time variant systems
- Time invariant systems
- Multiple input multiple output systems
- This gives idea about the internal state of the system

Concept



The output not only depends on the input applied to the system for $t > t_0$, but also on the initial conditions at time $t = t_0$

$$y(t) = y(t)\big|_{t=t0} + y(t)\big|_{t\geq t0}$$

$$= \int_{-\infty}^{t_0} y(t) + \int_{t_0}^{t} y(t)$$

$$= y(t_0) + \int_{0}^{t} y(t)$$

The term $y(t_0)$ is called the state of the system.

The variable that represents this state of the system is called the state variable

Definitions

State: The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time t = t_o (Initial condition), together with the knowledge of input for t ≥ t_0 , completely determines the behavior of the system for any time $t \geq t_0$.

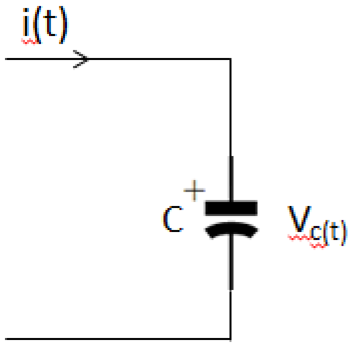
State Variables: A set of variables which describes the system at any time instant are called state variables

State vector: A vector whose elements are the state variables

State space: The n-dimensional space whose co-ordinate axes consists of the x_1 axis, x_2 axis,.... x_n axis, (where x_1 , x_2 ,..... x_n are state variables:) is called a state space.

Illustration

Consider the circuit shown in figure



$i(t) = C\,\frac{dv_c(t)}{dt}$

$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$

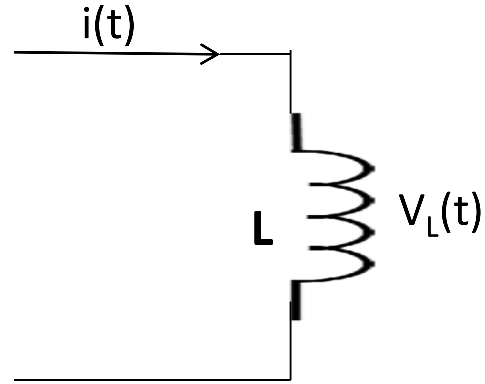
$= \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt + \frac{1}{C}\int_{t_0}^{t} i(t)dt$

$= V_c(t_0) + V_c(t)$

$V_c(t_0)$ = initial voltage across capacitor

The voltage across capacitor can be taken as a state variable

Consider the circuit shown in figure



$$V_L(t) = L\,\frac{di(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(t)dt$$

$$= \frac{1}{L}\int_{-\infty}^{t_0} V_L(t)\,dt + \frac{1}{L}\int_{t_0}^{t} V_L(t)\,dt$$

$$= V_L(t_0) + V_L(t)$$

$V_L(t_0)$ = initial voltage across Inductor

The current through inductor can be taken as a state variable

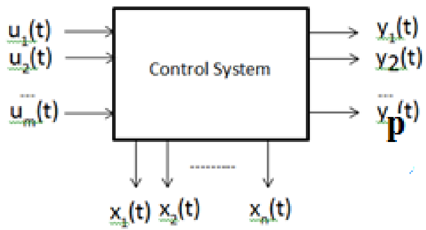
State Model

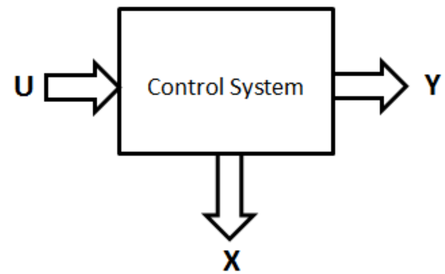
consider a multi-input & multi-output system is having

m inputs $u_1(t)\ ,\ u_2(t)\ ,\ldots\ldots u_m(t)$

p outputs $y_1(t)\ ,\ y_2(t)\ ,\ldots\ldots y_p(t)$

n state variables $x_1(t)\ ,\ x_2(t),\ldots\ldots x_n(t)$





The different variables may be represented by the vectors as shown below

Input vector U(t) = $\begin{bmatrix} u_1(t) \\ u_2(t) \\ .. \\ u_m(t) \end{bmatrix}$; Output vector Y(t) = $\begin{bmatrix} y_1(t) \\ y_2(t) \\ .. \\ y_p(t) \end{bmatrix}$

State variable vector X(t) = $\begin{bmatrix} x_1(t) \\ x_2(t) \\ .. \\ x_n(t) \end{bmatrix}$

State Equations

The state variable representation can be arranged in the form of n number of first order differential equations as shown below:

$$\frac{dx_1}{dt} = \dot{x}_1 = f1(x_1,x_2,....x_n\,;\,u_1,u_2.....u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f1(x_1,x_2,....x_n\,;\,u_1,u_2.....u_m)$$

……

……

$$\frac{dx_n}{dt} = \dot{x}_n = f1(x_1,x_2,....x_n\,;\,u_1,u_2.....u_m)$$

In vector notation, $\dot{X}(t)= f(X(t),U(t))$

Similarly the output vector $Y(t) = f(X(t),U(t))$

State Model of Linear System

The state model of a system consist of state equation and output equation.

The state equation of a system is a function of state variables and inputs.

For LTI systems, the first derivatives of state variables can be expressed as a linear combination of sate variables and inputs

$\dot{x}_1 = a_{11}x_1+a_{12}x_2+----+a_{1n}x_n + b_{11}u_1+b_{12}u_2+---+b_{1m}u_m$

$\dot{x}_2 = a_{21}x_1+a_{22}x_2+----+a_{2n}x_n + b_{21}u_1+b_{22}u_2+---+b_{2m}u_m$

$\dot{x}_n = a_{n1}x_1+a_{n2}x_2+----+a_{nn}x_n + b_{n1}u_1+b_{n2}u_2+---+b_{nm}u_m$

where the coefficients a_{ij} and b_{ij} are constants

In the matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ .. \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ .. \\ .. \\ u_m \end{bmatrix}$$

$\dot{X}(t) = A\,X(t) + B\,U(t)$……… state equation

where, A is state matrix of size (n×n)

B is the input matrix of size (n×m)

X(t) is the state vector of size (n×1)

U(t) is the input vector of size (m×1)

Output equation

The output at any time are functions of state variables and inputs.

output vector, $Y(t) = f(x(t), U(t))$

Hence the output variables can be expressed as a linear combination of state variables and inputs.

$y_1 = c_{11}x_1 + c_{12}x_2 + \text{----} + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \text{---} + d_{1m}u_m$

$y_2 = c_{21}x_1 + c_{22}x_2 + \text{----} + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \text{---} + d_{2m}u_m$

$y_p = c_{p1}x_1 + c_{p2}x_2 + \text{----} + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \text{---} + d_{pm}u_m$

where the coefficients c_{ij} and d_{ij} are constants

In the matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot\cdot \\ \cdot\cdot \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ \cdot\cdot \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ d_{21} & \cdots & d_{2m} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot\cdot \\ \cdot\cdot \\ u_m \end{bmatrix}$$

$Y(t) = C\,X(t) + D\,U(t)$.............. output equation

where, C is the output matrix of size $(p \times n)$

D is the transmission matrix of size $(p \times m)$

$X(t)$ is the state vector of size $(n \times 1)$

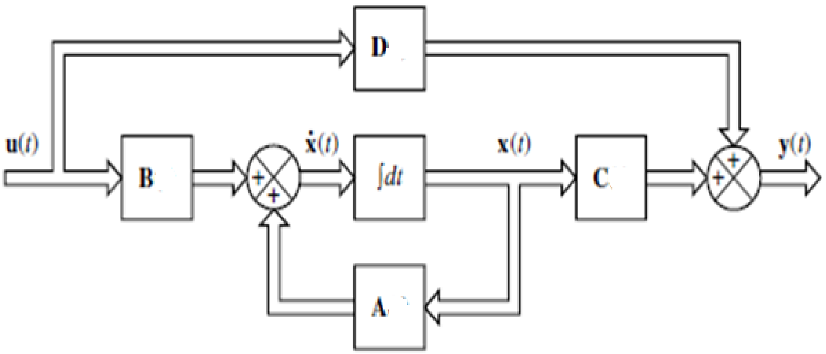
$Y(t)$ is the output vector of size $(p \times 1)$

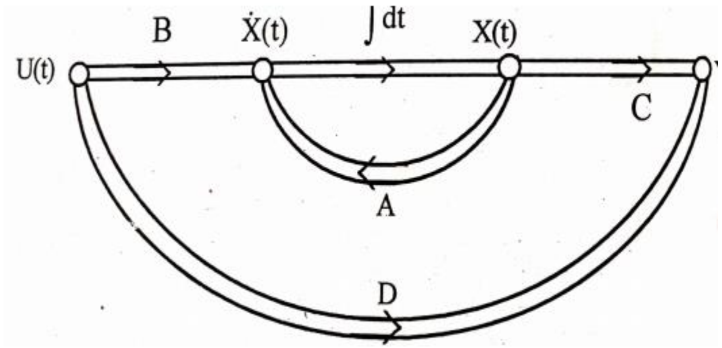
$U(t)$ is the input vector of size $(m \times 1)$

State Model

$\dot{X}(t) = A\,X(t) + B\,U(t)$ state equation

$Y(t) = C\,X(t) + D\,U(t)$ output equation





Selection of state variables

- The state variables of a system are not unique.
- There are many choices for a given system

Guide lines:

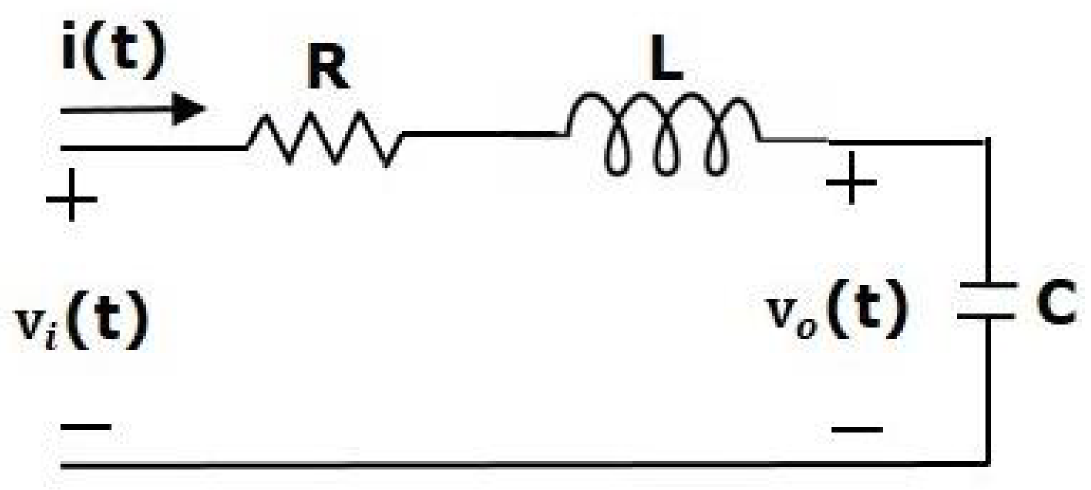
1. For a physical systems, the number of state variables needed to represent the system must be equal to the number of energy storing elements present in the system
2. If a system is represented by a linear constant coefficient differential equation, then the number of state variables needed to represent the system must be equal to the order of the differential equation
3. If a system is represented by a transfer function, then the number of sate variables needed to represent the system must be equal to the highest power of s in the denominator of the transfer function.

State space Representation using Physical variables

- In state-space modeling of the systems, the choice of sate variables is arbitrary.
- One of the possible choice is physical variables.
- The state equations are obtained from the differential equations governing the system

State Space Model

Consider the following series of the RLC circuit. It is having an input voltage $v_i(t)$ and the current flowing through the circuit is i(t).



- There are two storage elements (inductor and capacitor) in this circuit. So, the number of the state variables is equal to two.
- These state variables are the current flowing through the inductor, $i(t)$ and the voltage across capacitor, $v_c(t)$.
- From the circuit, the output voltage, $v_0(t)$ is equal to the voltage across capacitor, $v_c(t)$.

$Y(t) = v_0(t) = v_c(t)$

Apply KVL around the loop,

$$V_i(t) = R\, i(t) + L \frac{di(t)}{dt} + v_c(t)$$

$$\dot{i(t)} = \frac{di(t)}{dt} = -\frac{R}{L} i(t) - \frac{1}{L} v_c(t) + \frac{1}{L} V_i(t)$$

The voltage across the capacitor is

$$v_c(t) = \frac{1}{c} \int i(t)dt$$

Differentiate the equation with respect to time,

$$\dot{v_c(t)} = \frac{dv_c(t)}{dt} = \frac{i(t)}{c}$$

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$; Differential state vector, $\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix}$

Arrange the differential equations and output equation into standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$Y = [0 \quad 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

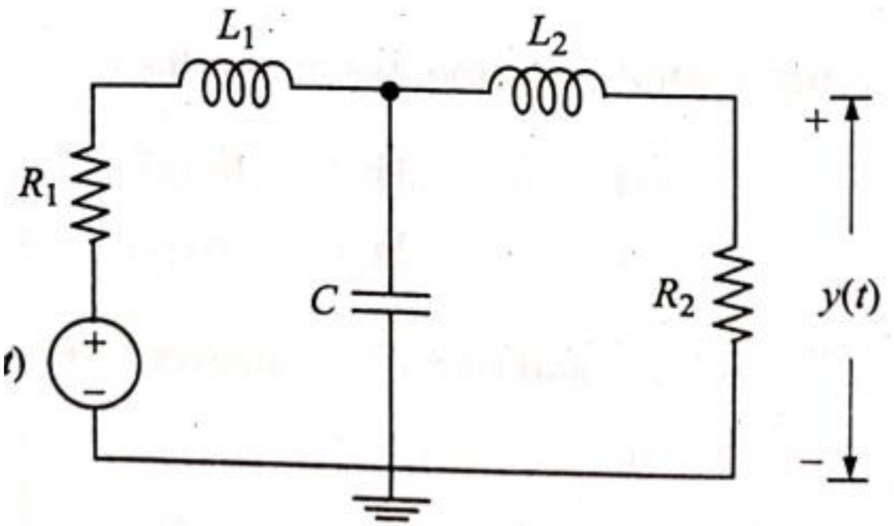
$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$Y(t) = C\,X(t) + D\,U(t)$$

Here $A = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$; $B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$; $C = [0 \quad 1]$; $D = [0]$

Problem

Represent the electrical circuit shown by a state model



Solution

Since there are three energy storing elements, choose three state variables to represent the systems

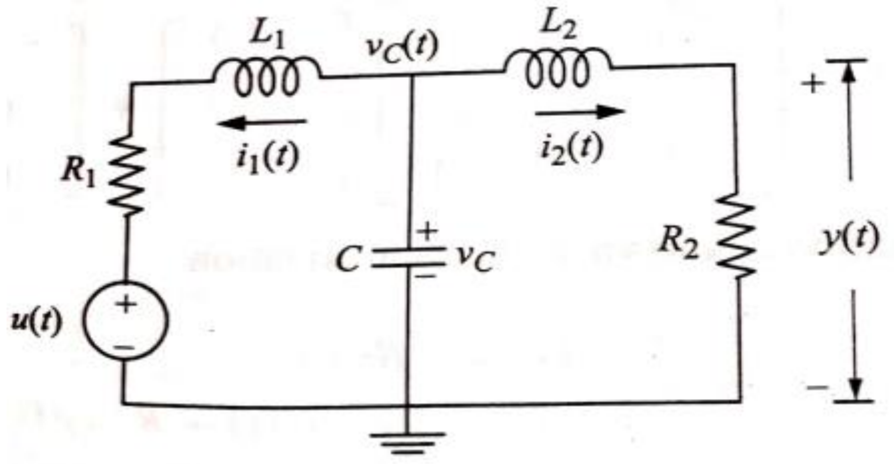
The current through the inductors i_1,i_2 and voltage across the capacitor v_c are taken as state variables

Let the three sate variables be x1, x2 and x3 be related to physical quantities as shown

Let, $i_1 = x_1,$

$i_2 = x_{2,}$

$v_c = x_3$



Applying KVL to loop 1,

$$L_1 \frac{di_1(t)}{dt} + R_1 i_1(t) + u(t) - v_c(t) = 0$$

$$\Longrightarrow \quad L_1 \frac{dx_1(t)}{dt} + R_1 x_1(t) + u(t) - x_3(t) = 0$$

$$\Longrightarrow \quad L_1 \dot{x}_1(t) + R_1 x_1(t) + u(t) - x_3(t) = 0$$

$$\Longrightarrow \quad \dot{x}_1(t) = -\frac{R_1}{L_1} x_1(t) + \frac{1}{L_1} x_3(t) - \frac{1}{L_1} u(t) \text{-------} (1)$$

Applying KVL to loop 2,

$$L_2 \frac{di_2(t)}{dt} + R_2 i_2(t) - v_c(t) = 0$$

$$\Longrightarrow \quad L_2 \frac{dx_2(t)}{dt} + R_2 x_2(t) - x_3(t) = 0$$

$$\Longrightarrow \quad L_2 \dot{x}_2(t) + R_2 x_2(t) - x_3(t) = 0$$

$$\Longrightarrow \quad \dot{x}_2(t) = -\frac{R_2}{L_2} x_2(t) + \frac{1}{L_2} x_3(t) \text{------------------} (2)$$

Applying KCL at node $v_c(t)$,

$$i_1(t) + i_2(t) + C\frac{dv_c(t)}{dt} = 0$$

$$\Longrightarrow \quad x_1(t) + x_2(t) + C\frac{dx_3(t)}{dt} = 0$$

$$\Longrightarrow \quad \dot{x}_3(t) = -\frac{1}{C}x_1(t) - \frac{1}{C}x_2(t) \text{ ---------------(3)}$$

Putting 1, 2and 3 in matrix form,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}-\frac{R_1}{L_1} & 0 & \frac{1}{L_1}\\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2}\\ -\frac{1}{C} & -\frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}-\frac{1}{L_1}\\ 0\\ 0\end{bmatrix}[u(t)]$$

This is State Equation

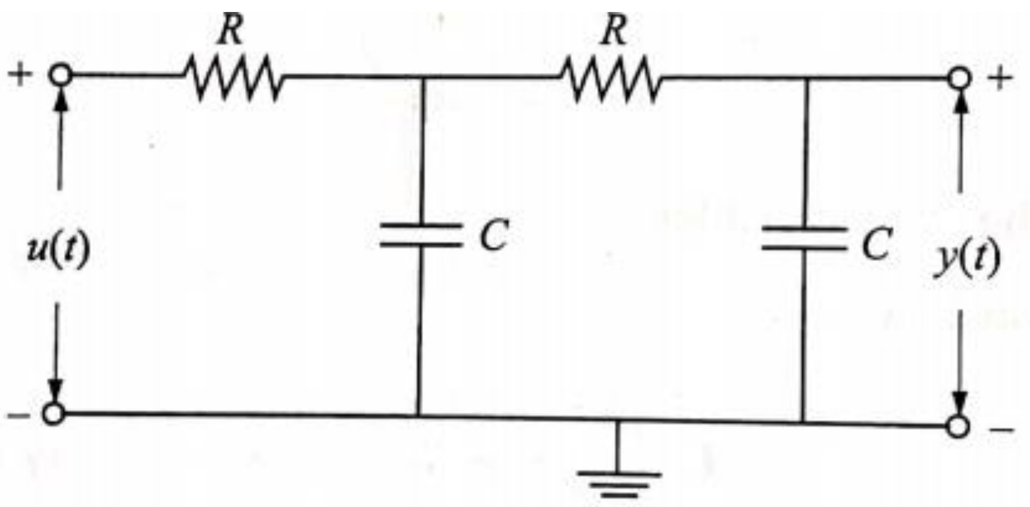
$$y(t) = R_2 i_2(t) = R_2 x_2(t)$$

$$y(t) = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

This is output equation

Problem

Obtain the state model for a system represented by an electrical system as shown in figure



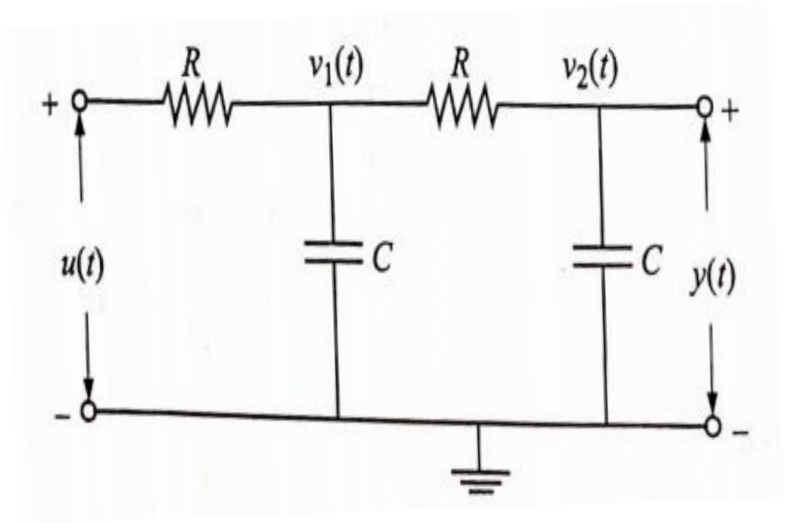
Solution

Since there are two energy storage elements present in the system, assume two state variables to describe the system behavior.

Let the two state variables be x_1 and x_2 be related to physical quantities as shown

Let $v_1(t) = x_1(t)$

$v_2(t) = x_2(t)$



Applying KCL at node $v_1(t)$,

$$\frac{v_1(t)-u(t)}{R} + C\frac{dv_1(t)}{dt} + \frac{v_1(t)-v_2(t)}{R} = 0$$

$$\Longrightarrow \quad \frac{x_1(t)-u(t)}{R} + C\frac{dx_1(t)}{dt} + \frac{x_1(t)-x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \frac{2x_1(t)}{R} - \frac{u(t)}{R} + C\,\dot{x}_1(t) - \frac{x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \dot{x}_1(t) = -\frac{2x_1(t)}{RC} + \frac{x_2(t)}{RC} + \frac{u(t)}{RC} \text{------------} (1)$$

Applying KCL at node $v_2(t)$,

$$C\,\frac{dv_2(t)}{dt} + \frac{v_2(t) - v_1(t)}{R} = 0$$

$$\Longrightarrow \quad C\,\frac{dx_2(t)}{dt} + \frac{x_2(t) - x_1(t)}{R} = 0$$

$$\Longrightarrow \quad C\,\dot{x}_2(t) - \frac{x_1(t)}{R} + \frac{x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \dot{x}_2(t) = \frac{x_1(t)}{RC} - \frac{x_2(t)}{RC} \qquad \text{----------(2)}$$

putting 1 and 2 in matrix form,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} [u(t)]$$

This is the state equation

The output of the circuit is given by

$$y(t) = v_2(t)$$

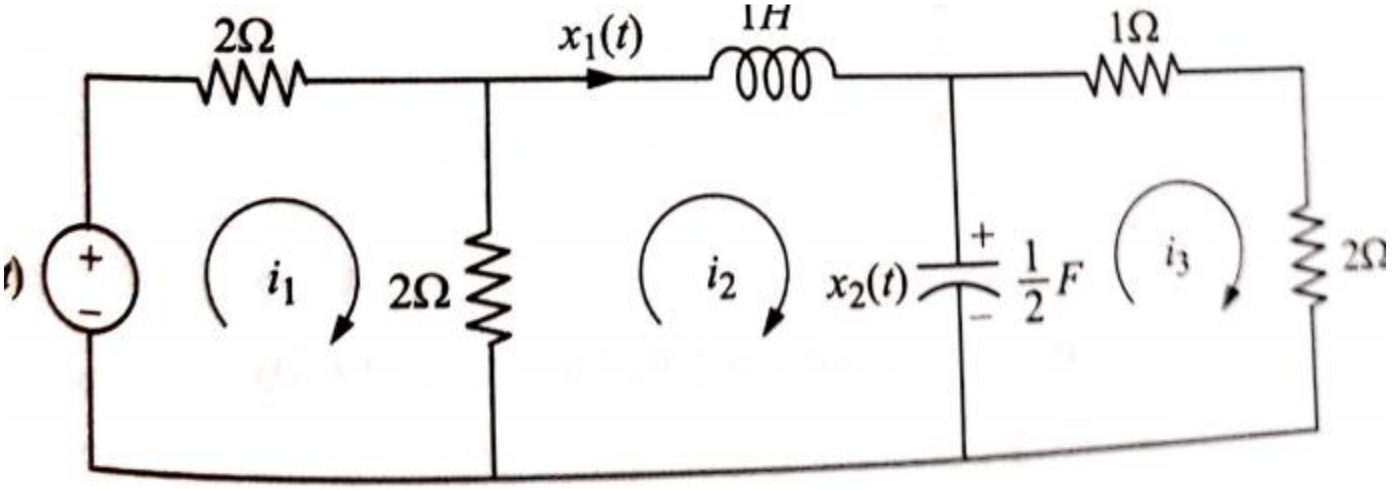
$$= x_2(t)$$

$$= [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This is the output equation

Problem

Represent the electrical network by a state equation



State representation using Phase variables

- The phase variables are defined as those particular state variables which are obtained from one of the system variables and its derivatives.
- Usually the variables used is the system output and the remaining state variables are then derivatives of the output.
- The state model using phase variables can be easily determined if the system model is already known in the differential equation or transfer function form.

Consider the following n^{th} order linear differential equation relating the output y(t) to the input u(t) of a system.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1}y}{dt^{n-1}} + a_2 \frac{d^{n-2}y}{dt^{n-2}} + \text{------} + a_{n-1}\frac{dy}{dt} + a_n y = u$$

Let us define the state variables as

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \frac{dx_1}{dt}$$

$$x_3 = \frac{d^2y}{dt^2} = \frac{d\dot{y}}{dt} = \frac{dx_2}{dt}$$

$$\vdots \;\; \vdots \;\; \vdots$$

$$x_n = \frac{d^{n-1}y}{dt^{n-1}} = \frac{dx_{n-1}}{dt}$$

From the above equations we can write

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots \;\; \vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n + a_1 x_n + \ldots\ldots + a_{n-1} x_2 + a_n x_1 = u$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \ldots\ldots\ldots - a_1 x_n + u$$

writing the above state equation in vector matrix form,

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & & 0 \\ \vdots & \ddots & \vdots \\ -an & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} [u]$$

Output equation can be written as

$$Y(t) = C\,X(t) = [1\ 0\ 0\ ..] \begin{bmatrix} x_1 \\ x_2 \\ \\ \dot{x}_n \end{bmatrix}$$

Problem

Construct a state model for a system characterized by the differential equation,

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$$

Also give the block diagram representation of the state model

Solution: Let us choose y and their derivatives as state variables. The system is governed by third order differential equation, so the number of state variables required are three.

Let the state variables x_1, x_2 and x_3 are related to phase variables as follows.

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \dot{x}_1$$

$$x_3 = \frac{d^2y}{dt^2} = \frac{dx_2}{dt} = \dot{x}_2$$

Put $y = x_1$, $\frac{dy}{dt} = x_2$, $\frac{d^2y}{dt^2} = x_3$ and $\frac{d^3y}{dt^3} = \dot{x}_3$ in the given equation

$$\therefore \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$$

$$\Longrightarrow \dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

Arranging the state equations in the matrix form,

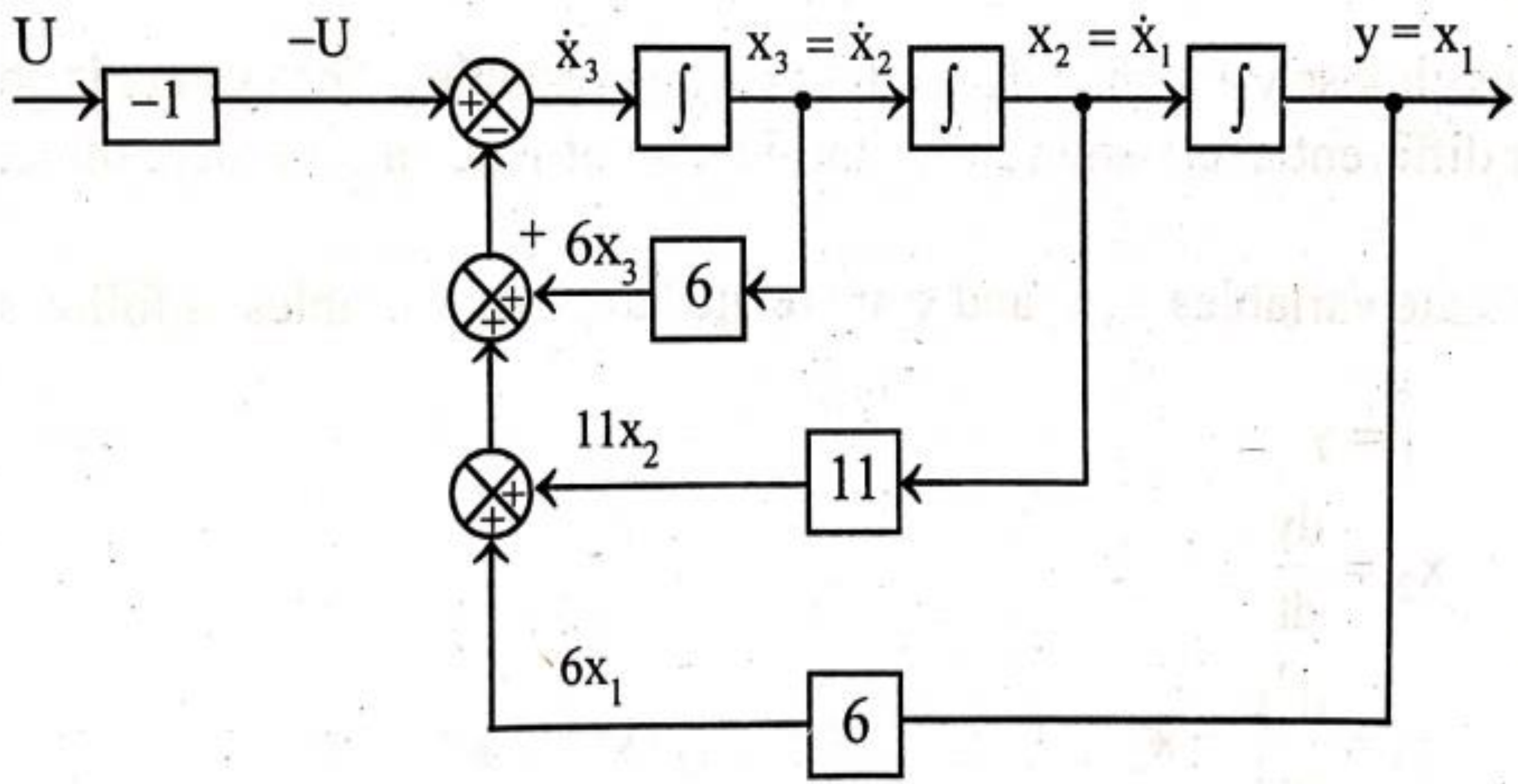
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u]$$

Here y = output

But $y = x_1$

$\therefore$ The output equation is , $y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The block diagram for the state model is



Problem

Represent the differential equation given below in a state model

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 7y = 2u(t)$$

Solution:

Since, the given equation is a third-order differential equation, choose three state variables to represent the system

Let $y(t) = x_1(t)$

$\dot{y}(t) = x_2(t)$

$\ddot{y}(t) = x_3(t)$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ are the state variables of the system.

$$y(t) = x_1(t)$$

$$\dot{y}(t) = x_2(t) = \dot{x}_1(t) \qquad \text{-----------(1)}$$

$$\ddot{y}(t) = x_3(t) = \dot{x}_2(t) \qquad \text{-----------(2)}$$

From the given diff. equation,

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 6\,\frac{dy}{dt} + 7y = 2u(t)$$

$$\dddot{y}\,(t) + \ddot{y}(t) + 6\dot{y}(t) + 7y = 2u(t)$$

$$\dot{x}_3(t) + x_3(t) + 6\,x_2(t) + 7\,x_1(t) = 2u(t)$$

$$\dot{x}_3(t) = -7\,x_1(t) - 6\,x_2(t) - x_3(t) + 2u(t) \quad \text{------}(3)$$

$$\dot{x}_2(t) = x_3(t) \quad \text{------}(2)$$

$$\dot{x}_1(t) = x_2(t) \quad \text{------}(1)$$

Putting the above equations in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

This is the state equation

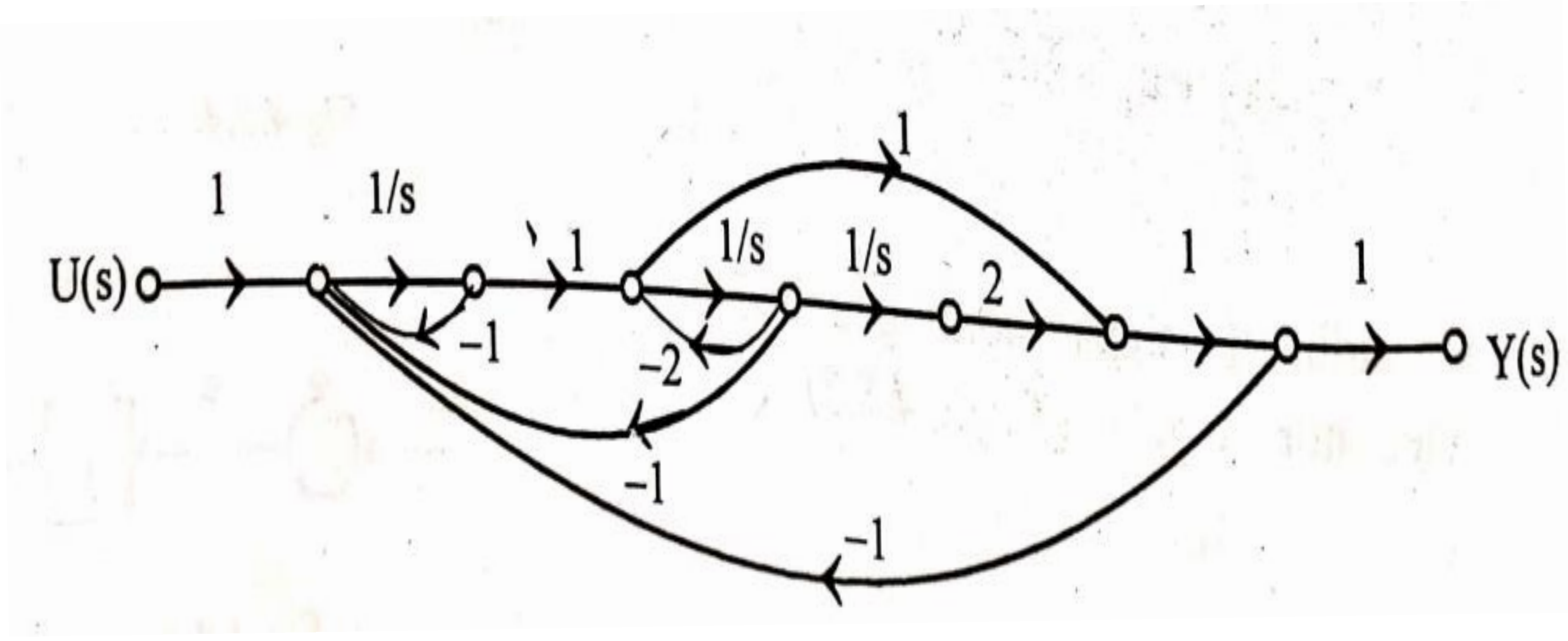
The output expression is $y(t) = x_1(t)$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This is the output equation

Problem

Obtain the state model for the signal flow graph given below:



Problem

Obtain the state model of the system whose transfer function is given by $\frac{Y(s)}{U(s)} = \frac{24}{s^3+9s^2+26s+24}$

Solution:

$$\frac{Y(s)}{U(s)} = \frac{24}{s^3+9s^2+26s+24}$$

Cross-multiplying yields

$$[s^3 + 9s^2 + 26s + 24]\, Y(s) = 24\, U(s)$$

$$s^3\, Y(s) + 9s^2\, Y(s) + 26sY(s) + 24\, Y(s) = 24U(s)$$

Taking inverse Laplace transforms,

$$\frac{d^3y(t)}{dt^3} + 9\,\frac{d^2y(t)}{dt^2} + 26\,\frac{dy(t)}{dt} + 24\, y(t) = 24u(t)$$

$$\dddot{y}(t) + 9\ddot{y}(t) + 26\dot{y}(t) + 24\, y(t) = 24u(t)$$

Choosing the state variables as successive derivatives

$x_1(t) = y(t)$

$x_2(t) = \dot{y}(t)$

$x_3(t) = \ddot{y}(t)$

$x_1(t) = y(t)$

$\dot{x}_1(t) = x_2(t) = \dot{y}(t)$ -----(1)

$\dot{x}_2(t) = x_3(t) = \ddot{y}(t)$ -----(2)

$\dot{x}_3(t) = \dddot{y}(t)$

$\dddot{y}(t) + 9\ddot{y}(t) + 26\dot{y}(t) + 24\, y(t) = 24u(t)$

$\dot{x}_3(t) + 9x_3(t) + 26x_2(t) + 24x_1(t) = 24u(t)$

$\dot{x}_3(t) = -\, 24x_1(t) - 26x_2(t) - \; 9x_3(t) + 24u(t)$------(3)

Putting equations 1, 2 and 3 in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} [u]$$

The output expression is $y(t) = x_1(t)$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matlab

Obtain the state model of the system whose transfer function is given by $\frac{Y(s)}{U(s)} = \frac{24}{s^3+9s^2+26s+24}$ using Matlab

```
>> num=[24];
>> den=[1 9 26 24];
>> [A,B,C,D] = tf2ss(num,den)
```

```
A =

    -9   -26   -24
     1     0     0
     0     1     0


B =

     1
     0
     0


C =

     0     0    24


D =

     0
```

Problem

Obtain the state model of the system whose transfer function is given by $\frac{Y(s)}{U(s)} = \frac{1}{s^2+s+1}$

Solution: $$\frac{Y(s)}{U(s)} = \frac{1}{s^2+s+1}$$

$$\Longrightarrow \quad (s^2 + s + 1)Y(s) = U(s)$$

$$s^2 Y(s) + s\,Y(s) + Y(s) = U(s)$$

Taking inverse Laplace transform on both sides,

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = u(t)$$

Let $y(t) = x_1$

$$\frac{dy(t)}{dt} = x_2 = \dot{x}_1 \text{ and } u(t) = u$$

Then the state equation is, $\dot{x}_2 = -x_1 - x_2 + u$

The output equation is, $y(t) = y = x_1$

The state space model is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem

Obtain the state model of the system whose transfer function is given by $\frac{s^2+7s+2}{s^3+9s^2+26s+24}$

Solution:

$$\frac{Y(s)}{U(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{C(s)} \times \frac{C(s)}{U(s)}$$

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2 \qquad \text{--------(1)}$$

$$\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24} \qquad \text{-------(2)}$$

Consider equation (2), $\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24}$

Cross-multiplying on both sides,

$[s^3 + 9s^2 + 26s + 24]\, C(s) = U(s)$

$s^3 C(s) + 9s^2 C(s) + 26s\, C(s) + 24\, C(s) = U(s)$

Taking inverse Laplace transform,

$$\frac{d^3c(t)}{dt^3} + 9\,\frac{d^2c(t)}{dt^2} + 26\,\frac{dc(t)}{dt} + 24\, c(t) = u(t)$$

$\dddot{c}(t) + 9\,\ddot{c}(t) + 26\,\dot{c}(t) + 24\, c(t) = u(t)$

$x_1(t) = c(t)$

$\dot{x}_1(t) = x_2(t) = \dot{c}(t)$ -----(3)

$\dot{x}_2(t) = x_3(t) = \ddot{c}(t)$ -----(4)

$\dot{x}_3(t) = \dddot{c}(t)$

$$\dddot{c}(t) + 9\ddot{c}(t) + 26\dot{c}(t) + 24c(t) = u(t)$$

$$\dot{x}_3(t) + 9\,x_3(t) + 26x_2(t) + 24x_1(t) = u(t)$$

$$\dot{x}_3(t) = -\,24x_1(t) - 26x_2(t) - 9\,x_3(t) + u(t) \quad \text{-----}(5)$$

Putting equations 3, 4 and 5 in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

Consider equation (1),

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2$$

$$Y(s) = [s^2 + 7s + 2]C(s)$$

$$Y(s) = s^2C(s) + 7sC(s) + 2C(s)$$

Taking inverse Laplace transform,

$$y(t) = \ddot{c}(t) + 7\dot{c}(t) + 2c(t)$$

$$y(t) = 2x_1(t) + 7x_2(t) + x_3(t)$$

$$y(t) = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matlab

Obtain the state model of the system whose transfer function is given by $\frac{s^2+7s+2}{s^3+9s^2+26s+24}$ using Matlab

```
>> num=[1 7 2];
>> den=[1 9 26 24];
>> [A,B,C,D] = tf2ss(num,den)

A =

    -9   -26   -24
     1     0     0
     0     1     0


B =

     1
     0
     0


C =

     1     7     2


D =

     0
```

Problem

A feedback system has a closed-loop transfer function $\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$

Solution:

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$$

By partial fraction expansion,

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+4)}$$

Solving for A, B and C

A = 3; B = - 4; C = 1

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{3}{(s+2)} - \frac{4}{(s+3)} + \frac{1}{(s+4)} \text{-----(1)}$$

$$= \frac{3}{s(1+2/s)} - \frac{4}{s(1+3/s)} + \frac{1}{s(1+4/s)}$$

$$= \frac{\frac{1}{s}}{(1+\frac{1}{s}*2)} \text{ x } 3 - \frac{\frac{1}{s}}{(1+\frac{1}{s}*3)} \text{ x } 4 + \frac{\frac{1}{s}}{(1+\frac{1}{s}*4)}$$

$$\therefore Y(s) = [\frac{\frac{1}{s}}{(1+\frac{1}{s}*2)} \text{ x } 3 - \frac{\frac{1}{s}}{(1+\frac{1}{s}*3)} \text{ x } 4 + \frac{\frac{1}{s}}{(1+\frac{1}{s}*4)}]u(s)$$

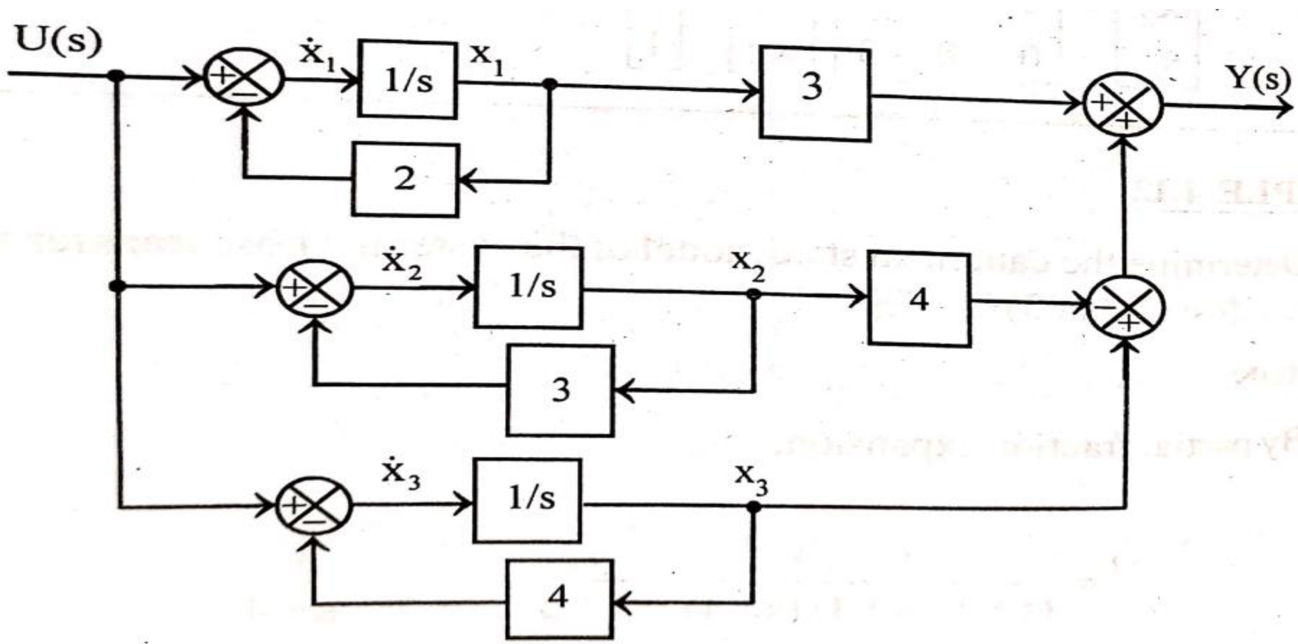
$$= [\frac{\frac{1}{s}}{(1+\frac{1}{s}*2)} \text{ x } 3]U(s) - [\frac{\frac{1}{s}}{(1+\frac{1}{s}*3)} \text{ x } 4]\, U(s) + [\frac{\frac{1}{s}}{(1+\frac{1}{s}*4)}]u(s)$$

The equation can be represented by the block diagram as shown

Assign state variables at the output of the integrators as shown.

At the input of the integrators , first derivative of the state variables are present.

The state equations are formed by adding all the incoming signals to the integrator and equating to the corresponding first derivative of state variables.



The state equations are

$$\dot{x}_1 = -2x_1 + u$$

$$\dot{x}_2 = -3x_2 + u$$

$$\dot{x}_3 = -4x_3 + u$$

The output equation is

$$y = 3x_1 - 4x_2 + x_3$$

The State model is given by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matlab

Find the state model for the transfer function $\frac{2(s+5)}{(s+2)(s+3)(s+4)}$ using Matlab

```
>> num=[2 10];
>> den=[1 9 26 24];
>> [A,B,C,D]=tf2ss(num,den)

A =

    -9   -26   -24
     1     0     0
     0     1     0


B =

     1
     0
     0


C =

     0     2    10


D =

     0
```

Solution of State Equation

S-Domain:

The State equation of n^{th} order system is given by,

$\dot{x}(t) = A\,X(t) + BU(t);$ $X(0) = X_0$ = initial condition vector

Taking Laplace transforms on both sides,

$S\,X(s) - X(0) = A\,X(s) + B\,U(s)$

$X(s)[sI-A] = X(0) + B\,U(s)$ where I = unit matrix

$X(s) = [sI-A]^{-1} X(0) + [SI-A]^{-1} B\,U(s)$ --------------(1)

Taking inverse Laplace transforms on both sides,

$X(t) = L^{-1}\,[sI-A]^{-1} X(0) + L^{-1}\,[SI-A]^{-1} B\,U(s)$

where $L^{-1}\,[sI-A]^{-1} = \emptyset(t)$ = state transition matrix

$[sI-A]^{-1} = \emptyset(s)$ = Resolvent matrix

The solution of state equation is,

$X(t) = \emptyset(t)\,X(0) + L^{-1}\,[\emptyset(s).\,B\,U(s)]$

The output equation is,

$$y(t) = C\,x(t) + D\,u(t)$$

Taking Laplace transforms on both sides,

$$Y(s) = C\,X(s) + D\,U(s)$$

From equation (1), $X(s) = [sI\text{-}A]^{-1} X(0) + [SI\text{-}A]^{-1} B\,U(s)$

$$= C\{[sI\text{-}A]^{-1} X(0) + [SI\text{-}A]^{-1} B\,U(s)\} + D\,U(s)$$

$$= C[sI\text{-}A]^{-1} X(0) + C\,[SI\text{-}A]^{-1} B\,U(s) + D\,U(s)$$

For zero initial conditions, $X(0) = 0$

$$\therefore Y(s) = C\,[SI\text{-}A]^{-1} B\,U(s) + D\,U(s)$$

$$= \{C\,[SI\text{-}A]^{-1} B + D\}\,U(s)$$

The transfer function $= \dfrac{Y(s)}{U(s)} = C\,[SI\text{-}A]^{-1} B + D$

Problem

A state variable description of a system is given by the matrix equation,

$$\dot{X} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$Y = [\,1 \quad 1\,]\,X$$

Find (i) The Transfer function

(ii) The State transition matrix

(iii) State diagram

Solution

The state model is given by

$$\dot{X} = A\,X + B\,U$$

$$Y = C\,X + D\,U$$

From the given problem,

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = [\,1 \quad 1\,]$$

(i) The transfer function = $\frac{Y(s)}{U(s)} = C\,[SI-A]^{-1}\,B + D$

Here D =0

$$\therefore \frac{Y(s)}{U(s)} = C\,[SI-A]^{-1}\,B$$

$$[sI\text{-}A] = s\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 1 & -2\end{bmatrix}$$

$$= \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}-1 & 0\\ 1 & -2\end{bmatrix}$$

$$= \begin{bmatrix}s+1 & 0\\ -1 & s+2\end{bmatrix}$$

$$[SI\text{-}A]^{-1} = \frac{\text{Adj } A}{\text{Det } A} = \begin{bmatrix}s+2 & 0\\ 1 & s+1\end{bmatrix}\frac{1}{(s+1)(s+2)}$$

$$\frac{Y(s)}{U(s)} = C\,[SI\text{-}A]^{-1}\,B$$

$$= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}s+2 & 0\\ 1 & s+1\end{bmatrix}\left\{\frac{1}{(s+1)(s+2)}\right\}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$= \left\{\frac{1}{(s+1)(s+2)}\right\}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}s+2 & 0\\ 1 & s+1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \left\{\frac{1}{(s+1)(s+2)}\right\} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 \\ 1 \end{bmatrix}$$

$$= \left\{\frac{1}{(s+1)(s+2)}\right\} [s+3]$$

$$= \frac{s+3}{(s+1)(s+2)}$$

(ii) State transition matrix = $\emptyset(t) = L^{-1} [sI-A]^{-1}$

$$[SI-A]^{-1} = \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} \frac{1}{(s+1)(s+2)}$$

$$= \begin{bmatrix} \frac{s+2}{(s+1)(s+2)} & \frac{0}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+1}{(s+1)(s+2)} \end{bmatrix}$$

$$[SI-A]^{-1} = \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

$$\varnothing(t) = L^{-1}\,[sI-A]^{-1}$$

$$= L^{-1}\begin{bmatrix} \frac{1}{(s+1)} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{1}{(s+1)} & 0 \\ \frac{1}{(s+1)} - \frac{1}{(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix}$$

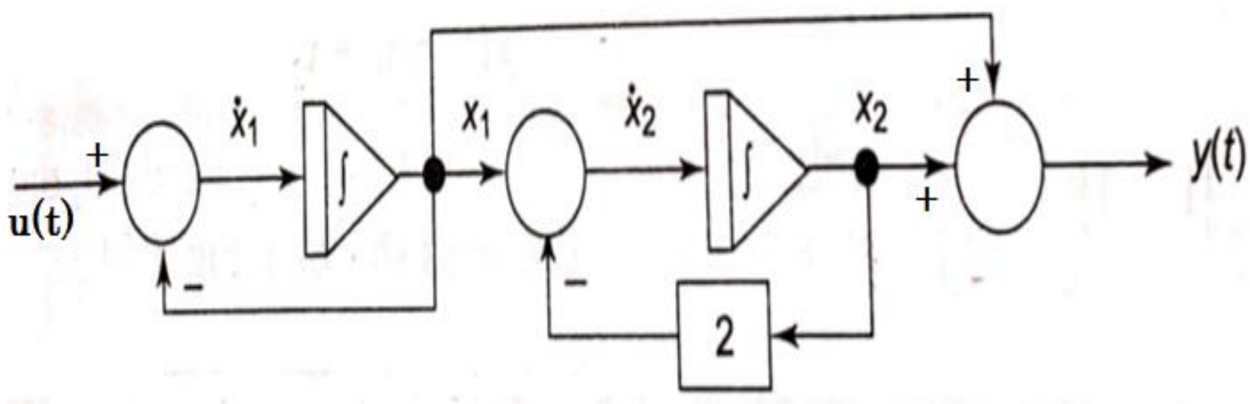
(iii) State Diagram:

The state equation is,

$$\dot{x}_1 = -x_1 + u(t)$$

$$\dot{x}_2 = x_1 - 2x_2$$

The output equation is $y = x_1 + x_2$



Matlab

Find the transfer function using Matlab

```
>> A=[-1,0; 1, -2];
>> B=[1; 0];
>> C=[1 1];
>> D=[0];
>> [num,den]= ss2tf(A,B,C,D)

num =

     0     1     3


den =

     1     3     2

>> g=tf(num,den)

g =

      s + 3
  -------------
  s^2 + 3 s + 2

Continuous-time transfer function.
```

Problem

The state equation of a LTI system is given as

$$\dot{x} = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \text{ and } y = [1 \quad 1]\, X$$

Determine (i) State transition matrix

(ii) The transfer function

(iii) State diagram

Solution

From the given system,

$$A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

(i) The State transition matrix ,

$$\emptyset(t) = L^{-1}\,[sI-A]^{-1}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} s & -5 \\ 1 & s+2 \end{bmatrix}$$

$$[sI-A]^{-1} = \begin{bmatrix} s+2 & 5 \\ -1 & s \end{bmatrix} \frac{1}{s(s+2)+5} = \begin{bmatrix} \frac{s+2}{s(s+2)+5} & \frac{5}{s(s+2)+5} \\ \frac{-1}{s(s+2)+5} & \frac{s}{s(s+2)+5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+1+1}{(s+1)^2+2^2} & \frac{5}{(s+1)^2+22} \\ \frac{-1}{(s+1)^2+22} & \frac{s}{(s+1)^2+22} \end{bmatrix}$$

$\emptyset(t) = L^{-1}\,[sI-A]-1$

$$= L^{-1} \begin{bmatrix} \frac{s+1+1}{(s+1)^2+22} & \frac{5}{(s+1)^2+22} \\ \frac{-1}{(s+1)^2+22} & \frac{s+1-1}{(s+1)^2+22} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\cos 2t + \frac{1}{2}\, e^{-t}\sin 2t & \frac{5}{2}\, e^{-t}\sin 2t \\ -\frac{1}{2}\, e^{-t}\sin 2t & e^{-t}\cos 2t - \frac{1}{2}\, e^{-t}\sin 2t \end{bmatrix}$$

(ii) The transfer function

$$\frac{Y(s)}{U(s)} = C\,[SI\text{-}A]^{-1}\,B$$

$$= [1 \quad 1] \begin{bmatrix} s+2 & 5 \\ -1 & s \end{bmatrix} \frac{1}{s(s+2)+5} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s+2)+5} [1 \quad 1] \begin{bmatrix} S+7 \\ S-1 \end{bmatrix}$$

$$= \frac{2S+6}{s(s+2)+5}$$

Matlab

Find the transfer function using Matlab

```
>> A=[0 5;-1 -2]

A =

     0     5
    -1    -2

>> B=[1;1]

B =

     1
     1

>> C=[1 1]

C =

     1     1

>> D=[0]

D =

     0
```

```
>> [num,den]=ss2tf(A,B,C,D)

num =

     0     2     6


den =

    1.0000    2.0000    5.0000

>> g=tf(num,den)

g =

     2 s + 6
  -------------
  s^2 + 2 s + 5

Continuous-time transfer function.
```

Problem

Find the transfer function of a state model of a system given by ,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} X$

Solution

From the given system,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[sI\text{-}A] = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$\therefore [sI-A]^{-1} = \begin{bmatrix} (s+2)(s+1) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1}$$

The transfer function of the system is given by

$$\frac{Y(s)}{U(s)} = C\,[SI-A]^{-1}\,B$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (s+2)(s+1) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ s(s+3) & s \\ -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1}$$

$$= \begin{bmatrix} s+3 & 1 \\ -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1}$$

Matlab

Find the transfer function using Matlab

```
>> A=[0 1 0;0 0 1;-1 -2 -3];
>> B=[0 0;0 1;1 0];
>> C=[1 0 0;0 0 1];
>> D=[0 0;0 0];
>> [num,den]=ss2tf(A,B,C,D,1)

num =

     0     0     0     1
     0     1     0     0


den =

    1.0000    3.0000    2.0000    1.0000
```

Problem

A linear time-invariant system is characterized by state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute the solution of the state equation , assuming the initial vector $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

From the given system, A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

The solution of state equation is,

$X(t) = L^{-1}\,[sI\text{-}A]^{-1}\,X(0) + L^{-1}\,[SI\text{-}A]^{-1}\,B\,U(s)$

Here U= 0

$\therefore X(t) = L^{-1}\,[sI\text{-}A]^{-1}\,X(0)$

$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$

$[sI\text{-}A]^{-1} = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} \frac{1}{(s-1)^2}$

$$[sI-A]^{-1} = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} \frac{1}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$X(t) = L^{-1} [sI-A]^{-1} X(0)$$

$$= L^{-1} \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

Solution of state equation (Time Domain)

$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad X(0) = x_0$

$\dot{x}(t) - A\,x(t) = B\,u(t)$

pre-multiplying both sides by e^{-At}

$e^{-At}[\dot{x}(t) - A\,x(t)] = e^{-At}\,B\,u(t)$ ----------------(1)

Consider, $\frac{d}{dt}\{e^{-At}\,x(t)\} = e^{-At}\,\dot{x}(t) - A\,e^{-At}\,x(t)]$

$= e^{-At}[\dot{x}(t) - A\,x(t)]$

$\therefore$ equation (1) $= \frac{d}{dt}\{e^{-At}\,x(t)\} = e^{-At}\,B\,u(t)$

Integrating with respect to t

$$\int_0^t \frac{d}{dt}\{e^{-At}\,x(t)\}\,dt = \int_0^t [e^{-A\tau}\,B\,u(\tau)]d\tau$$

$$e^{-At}\,x(t) - x(0) = \int_0^t [e^{-A\tau}\,B\,u(\tau)]d\tau$$

Pre-multiplying both sides by e^{At},

$$e^{At}\,[e^{-At}\,x(t) - x(0)] = e^{At}[\int_0^{t} e^{-A\tau}\, B\, u(\tau)d\tau]$$

$$x(t) = e^{At}[x(0) + \int_0^{t} e^{-A\tau}\, B\, u(\tau)d\tau]$$

$$= e^{At}\, x(0) + \int_0^{t}[e^{A(t-\tau)}\, B\, u(\tau)]d\tau$$

$$x(t) = \emptyset(t)\, x(0) + \int_0^{t} \emptyset(t-\tau)\, B\, u(\tau)\, d\tau$$

if the initial time is $t = t_0$, the solution of state equation becomes,

$$x(t) = \emptyset(t - t_0)\, x(t_0) + \int_0^{t} \emptyset(t-\tau)\, B\, u(\tau)d\tau$$

Properties of state transition matrix

$$\emptyset(t) = e^{At} = L^{-1}\,[sI-A]^{-1}$$

1. $\emptyset(0) = I$
2. $\emptyset^{-1}(t) = \emptyset(-t)$
3. $\emptyset(t_2-t_1)\ \emptyset(t_1-t_0) = \emptyset(t_2-t_0)$ for any t_2, t_1, t_0
4. $[\emptyset(t)]^k = \emptyset(kt)$
5. $\emptyset(t_1+t_2) = \emptyset(t_1)\ \emptyset(t_2) = \emptyset(t_2)\ \emptyset(t_1)$

Problem

Compute the State transition matrix by infinite series method $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Solution: For the given system matrix A, the state transition matrix is,

$$\emptyset\,(t) = e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \text{------}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$A^2 = A.\,A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

$$\emptyset(t) = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} t + \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} \frac{t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2} + \frac{t^3}{3} + \cdots & t - t^2 + \frac{t^3}{2} + \cdots \\ -t + t^2 - \frac{t^3}{2} + \cdots & 1 - 2t + \frac{3t^2}{2} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Problem

Find the state transition matrix by infinite series method for the system matrix A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Solution: For the given system matrix A, the state transition matrix is,

$$\emptyset\,(t) = e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \text{------}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\emptyset(t) = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + ---$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \frac{t^3}{3!} + ---$$

$$= \begin{bmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & t + t^2 + \frac{t^3}{2} + \cdots \\ 0 & 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + .. \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

References

1. Control Engineering by Nagrath & Gopal, New Age International Publishers
2. Engineering control systems - Norman S. Nise, John WILEY & sons , fifth Edition
3. Modern control Engineering-Ogata, Prentice Hall
4. Automatic Control Systems- B.C Kuo, John Wiley and Sons

Controllability and Observability

- The concept of controllability and observability were introduced by Kalman in 1960.

- They play an important role in the design of control systems in state space.

- The conditions of controllability and observability may govern the existence of a complete solution to the control system design problem.

- The solution of the problem may not exist if the system is not controllable.

Concepts of controllability and observability

Controllability:

- The controllability verifies the usefulness of a state variables. In the controllability test we can find, whether the state variable can be controlled to achieve the desired output.

Definition for controllability:

- A system is said to be completely state controllable if it is possible to transfer the system state from an initial state $X(t_0)$ to any other desired state $X(t_d)$ in specified finite time by a control vector $U(t)$.
- **The controllability of a state model can be tested by Kalman's and Gilbert's test.**

Gilbert's method of testing controllability:

Case(i): When the system matrix has distinct Eigen values

- In this case the system matrix can be diagonalized and the state model can be converted to canonical form.

Consider the state model of the system,

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

- The state model can be converted to canonical form by a transformation, X=MZ,
- Where M is the modal matrix and Z is the transformed state variable vector.

$$\dot{Z} = \Delta Z + \hat{B}U \quad Y = \check{C}Z + DU$$

The transformed state model is given by

where
$$\Delta = M^{-1}AM$$
$$\check{B} = M^{-1}B$$
$$\check{C} = CM$$

- In this case the necessary and sufficient condition for complete controllability is that, **the matrix $\tilde{B}$ must have no rows with all zeros.** If any row of the matrix $\tilde{B}$ **is zero** then the corresponding state variable is **uncontrollable.**

Case(ii): When the system matrix has repeated Eigen values

- In this case, the system matrix **cannot be diagonalized** but can be transferred to **Jordan canonical form**.

Consider the state model of the system,

$$\dot{X} = AX + BU \qquad Y = CX + DU$$

The state model can be transferred to Jordan canonical form by a transformation, X=MZ ,Where M is the modal matrix and Z is the transformed state variable vector.

The transformed state model is given by,

$$\dot{Z} = JZ + \hat{B}U \qquad Y = \check{C}Z + DU$$

$$J = M^{-1}AM$$

$$\check{B} = M^{-1}B \qquad \check{C} = CM$$

- In this case, the system is **completely controllable** if the elements of any row of $\tilde{B}$ that correspond to the last row of each Jordan **block are not zero** and the rows corresponding to other state **variables must not have all zeros.**

Kalman's method of testing controllability:

Consider a system with state equation, $\dot{X} = AX + BU$. For this system, a composite matrix, Q_c can be formed such that,

$$Q_C = [B\ AB\ A^2B \dots \dots \dots . A^{n-1}\ B]$$

Where n is the order of the system (n is also equal to number of state variables)

- **In this case the system is completely state controllable if the rank of the composite matrix, Q_c is in n. If**

$|Q_c| \neq 0$, *then rank of* $Q_c = n$ *and the system is completely state controllable*

Condition for complete state controllability in the s-plane:

A necessary and sufficient condition for complete state controllability

is that no cancellation of poles and zeros occurs in the transfer function of the system. If cancellation occurs then the system cannot be controlled in the direction of the cancelled mode.

Observability:

- In observability test we can find whether the state variable is observable or measurable. The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable ones in the minimum possible length of time.

Definition for Observability :

- A system is said to be completely observable if every state X(t) can be completely identified by measurements of the output Y(t) over a finite time interval.

Gilbert's method of testing observability:

The state model can be converted to a canonical or Jordan canonical form by a transformation, X=MZ

$$\dot{Z} = \Delta Z + \hat{B}U \qquad \dot{Z} = JZ + \hat{B}U$$

$$Y = \check{C}Z + DU \qquad Y = \check{C}Z + DU$$

The necessary and sufficient condition for complete observability is that none of the columns of the matrix $\check{C}$ be zero. If any of the column's of $\check{C}$ has all zeros then the corresponding state variable is not observable.

Kalman's method of testing observability: $\dot{X} = AX + BU \quad Y = CX + DU$

Consider a system with state model,

For this system, a composite matrix, Q_0 can be formed such that,

$$Q_0 = [C^T \;\; A^T C^T \;\; (A^T)^2 C^T \;\; \ldots\ldots\ldots (A^T)^{n-1} C^T \;]$$

Where n is the order of the system (n is also equal to number of state variables)

In this case the system is completely observable if the rank of composition matrix, Q_0 is n.

Condition for complete state observability in the s-plane:

A necessary and sufficient condition for complete state observability is that no cancellation of poles and zeros occurs in the transfer function of the system. If cancellation mode cannot be observed in

PROBLEM: Determine whether the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 3 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$$

is controllable.

ANSWER: Controllable

Use MATLAB, the Control System Toolbox, and the following statements to solve above Exercise

```
A=[1 1 2
0 1 5
0 3 4]
B=[2;1;1]
Cm=ctrb(A,B)
Rank=rank(Cm)
```

PROBLEM: Determine whether the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -2 & -1 & -3 \\ 0 & -2 & 1 \\ -7 & -8 & -9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{x} = [4 \quad 6 \quad 8]\,\mathbf{x}$$

is observable.

ANSWER: Observable

Use MATLAB, the Control System Toolbox, and the following statements to solve above Exercise.

```
A=[2 1 3
0 2 1
7 8 9]
C=[4 6 8]
Om=obsv(A,C)
Rank=rank(Om)
```

Concept of Eigen Values and Eigen Vectors

The roots of characteristic equation that we have described above are known as eigen values of matrix A.

Now there are some properties related to eigen values and these properties are written below-

1. Any square matrix A and its transpose A^T have the same eigen values.
2. Sum of eigen values of any matrix A is equal to the trace of the matrix A.
3. Product of the eigen values of any matrix A is equal to the determinant of the matrix A.
4. If we multiply a scalar quantity to matrix A then the eigen values are also get multiplied by the same value of scalar.
5. If we inverse the given matrix A then its eigen values are also get inverses.
6. If all the elements of the matrix are real then the eigen values corresponding to that matrix are either real or exists in complex conjugate pair.

Eigen Vectors

Any non zero vector m_i that satisfies the matrix equation $(\lambda_i I - A)m_i = 0$ is called the eigen vector of A associated with the eigen value λ_i,. Whereλ_i, i = 1, 2, 3, ……..n denotes the i^{th} eigen values of A.

This eigen vector may be obtained by taking cofactors of matrix $(\lambda_i I - A)$ along any row & transposing that row of cofactors.

Diagonalization

Let $m_1, m_2, \ldots\ldots.. m_n$ be the eigenvectors corresponding to the eigen value $\lambda_1, \lambda_2, \ldots\ldots.. \lambda_n$ respectively.

Then $M = [m_1 \ \vdots \ m_2 \ \vdots \cdots \ \vdots \ m_n]$ is called diagonalizing or **modal matrix** of A.

Consider the n[th] order MIMO state model

$$\dot{X}(t) = AX(t) + BU(t)$$

$$Y(t) = CX(t) + DU(t)$$

System matrix A is non diagonal, so let us define a new state vector V(t) such that X(t)=MV(t).

Under this assumption original state model modifies to

$$\dot{V}(t) = \tilde{A}V(t) + \tilde{B}U(t)$$

$$Y(t) = \check{C}V(t) + DU(t)$$

Where $\boldsymbol{\tilde{A} = M^{-1}AM} = diagonal\ matrix,\qquad \boldsymbol{\tilde{B} = M^{-1}B},\qquad \boldsymbol{\tilde{C} = CM}$

The above transformed state model is in canonical state model. The transformation described above is called similarity transformation.

If the system matrix A is in companion form & if all its n eigen values are distinct, then modal matrix will be special matrix called the **Vander Monde matrix.**

$$Vander\ Monde\ matrix\ V = \begin{bmatrix} 1 & 1 & 1 \ldots\ldots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \ldots\ldots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \lambda_3^{n-2} \ldots\ldots & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} \ldots\ldots. & \lambda_n^{n-1} \end{bmatrix}_{n\times n}$$

Problem1

Example 12.13 Consider the system defined by where,

$$\dot{x} = A\mathbf{x} + B\boldsymbol{u}$$
$$y = \boldsymbol{CA}$$
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 10 & 5 & 1 \end{bmatrix}$$

Check the system for (a) complete state controllability and (b) complete observability.

Problem1

Solution
a) Test for complete state controllability

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -12 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \\ 61 \end{bmatrix}$$

Problem1

So the controllability matrix Q_c is given by

$$Q_c = \begin{bmatrix} B: & AB: & A^2B \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{bmatrix}$$

Now,

$$|Q_d| = -84 \neq 0$$

So the rank of matrix Q_c is equal to its order, that is, 3 . This indicates that according to Kalman's test the system is completely state controllable.

Problem1

b) Test for complete observability

$$C^* = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$A^* C^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix}$$

$$(A^*)^2 C^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} -6 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}$$

Problem1

So the observability matrix Q_0 is given by

$$Q_0 = \left[C^* : A^* C^* : (A^*)^2 C^*\right]$$
$$= \begin{bmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{bmatrix}$$

Now,

$$|Q_0| = 96 \neq 0$$

So, the rank of matrix Q_0 is equal to its order, that is, 3 . This indicates that due to Kalman the system is completely observable.

Problem2

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Obtain the transfer function of the system.

Problem2

Solution

The given system may be written as where,

$$\dot{x} = Ax + Bu$$
$$y = Cx + du$$
$$A = \begin{bmatrix} -4 & -1 \\ 3 & -1 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$d = 0$$

$\therefore \quad (sI - A) = \begin{bmatrix} s+4 & 1 \\ -3 & s+1 \end{bmatrix}$ Thus the adjoint of matrix $(sI - A)$ is given by Adj

Problem2

$$(sI - A) = \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix}$$

$$|sI - A| = (s+4)(s+1) + 3$$

$$= s^2 + 5s + 7$$

Also, We may write the transfer function as

$$\frac{Y(s)}{U(s)} = \frac{C[adj(sI - A)]B}{|(sI - A)|} + d$$

Now, $C[\text{adj}(sI - A)]B$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 3 & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s \\ s+7 \end{bmatrix} = s$$

$$\frac{Y(s)}{U(s)} = \frac{s}{s^2 + 5s + 7}$$

Problem3

The state model of a system is given by where,

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

and Obtain a diagonal canonical form of the state model by a suitable transformation matrix. Solution
The characteristic equation of the matrix ' A ' is given by or,

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix} = 0$$

Problem3

or,

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

or,

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

$\therefore \lambda_1 = -1, \therefore \lambda_2 = -2$ and $\lambda_3 = -3$ are the three distinct eigenvalues of matrix ' A '. So the Vander Monde transformation matrix can be written as

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

Problem3

$$\therefore \quad P^{-1} = \frac{1}{|P|}(adj\ P) = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix}$$

$$\therefore \quad P^{-1}AP = P^{-1}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 4 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -2 & -3 \\ 1 & 4 & 9 \\ -1 & -8 & -27 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$P^{-1}B = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

$$CP = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Applying the transformation $x = Pz$ we get the diagonal canonical form of the state model as

$$\dot{z} = (P^{-1}AP)z + (P^{-1}B)u$$

$$y = (CP)z$$

or,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$