UNIT – III
COUPLED CIRCUITS & RESONANCE

Magnetically Coupled Circuits

When the two circuits are placed very close to each other such that a magnetic flux produced by one circuit links with both the circuits, then the two circuits are said to be **Magnetically Coupled Circuits.**

A wire of certain length, when twisted into coil becomes a basic inductor. If a current is made to pass through an inductor, an electromagnetic field is developed. A change in the magnitude of the current, changes the electromagnetic field and hence induces a voltage in coil according to Faraday's law of electromagnetic induction.

When two or more coils are placed very close to each other, then the current in one coil affects other coils by inducing voltage in them. Such coils are said to be **mutually coupled coils**. Such induced voltages in the coils are functions of the self inductances of the coils and mutual inductance between them. Let us study concept of self induced e.m.f. and mutually induced e.m.f.

Self inductance:

Consider a coil having N turns carrying current i as shown in the Fig. 2.1.

Due to the current flow, the flux ϕ is produced in the coil. The flux is measured in Wb (weber). The flux produced by the coil links with the coil itself. Thus the total flux linkage of the coil will be ($N\phi$) Wb-turns. If the current flowing through the coil changes, the flux produced in the coil also changes and hence flux linkage also changes.

According to Faraday's law, due to the rate of change of flux linkages, there will be induced e.m.f. in the coil. This phenomenon is called **self induction**. The e.m.f. or voltage induced in the coil due to the change of its own flux linked with it, is called **self induced e.m.f.**

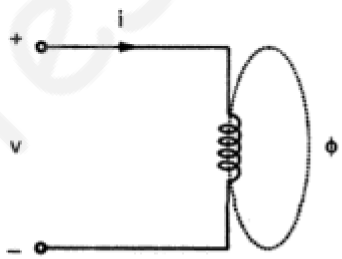


Fig. 2.1

According to Lenz's law the direction of this induced e.m.f. will be so as to oppose the cause producing it. The cause is the current I hence the self induced e.m.f. will try to set up a current which is in opposite direction to that of current I. When current is increased, self induced e.m.f. reduces the current tries to keep it to its original value. If current is decreased, self induced e.m.f. increases the current and tries to maintain it back to its original value. **So any change in current through coil is opposed by the coil.**

This property of the coil which opposes any change in the current passing through it is called **Self Inductance or Only Inductance.**

From the Faraday's law of electromagnetic induction, self induced e.m.f. can be expressed as

$$v = -N\frac{d\phi}{dt}$$

Negative sign indicates that direction of this e.m.f. is opposing change in current due to which it exists.

The flux can be expressed as,

$$\phi = (\text{Flux/ Ampere}) \times \text{Ampere} = \frac{\phi}{I} \times I$$

Now for a circuit, as long as permeability 'μ' is constant, ratio of flux to current (i.e. B/H) remains constant.

Rate of change of flux $= \frac{\phi}{I} \times$ Rate of change of current

$$\therefore \quad \frac{d\phi}{dt} = \frac{\phi}{I} \cdot \frac{dI}{dt}$$

$$v = -N \cdot \frac{\phi}{I} \cdot \frac{dI}{dt}$$

$$v = -\left(\frac{N\phi}{I}\right)\frac{dI}{dt}$$

The constant $\frac{N\phi}{I}$ in this expression is nothing but the quantitative measure of the property due to which coil opposes any change in current.

So this constant $\frac{N\phi}{I}$ is called **coefficient of self inductance and denoted by 'L'.**

$$\therefore \quad \boxed{L = \frac{N\phi}{I}}$$

It can be defined as flux linkages per ampere current in it. Its unit is **henry (H).**

A circuit possesses **a self inductance of 1 H** when a current of 1 A through it produces flux linkages of 1 Wb-turn in it.

$$\therefore \quad \boxed{v = -L\frac{dI}{dt} \quad \text{volts}}$$

Expressions for Coefficient of Self Inductance (L)

$$L = \frac{N\phi}{I}$$

But $$\phi = \frac{\text{M.M.F.}}{\text{Reluctance}} = \frac{NI}{S}$$

$$\therefore \quad L = \frac{N \cdot NI}{I \cdot S}$$

$$\therefore \quad \boxed{L = \frac{N^2}{S} \quad \text{henries}}$$

Now $$S = \frac{l}{\mu a}$$

$$L = \frac{N^2}{\left(\frac{l}{\mu a}\right)}$$

$$\therefore \quad \boxed{L = \frac{N^2 \mu a}{l} = \frac{N^2 \mu_0 \mu_r a}{l} \quad \text{henries}}$$

where l = Length of magnetic circuit

a = Area of cross-section of magnetic circuit through which flux is passing.

Example 1 : *If a coil has 500 turns is linked with a flux of 50 mWb, when carrying a current of 125 A. Calculate the inductance of the coil. If this current is reduced to zero uniformly in 0.1 sec, calculate the self induced e.m.f. in the coil.*

Solution : The inductance is given by,

$$L = \frac{N\phi}{I}$$

where $N = 500, \quad \phi = 50 \text{ mWb} = 50\times10^{-3} \text{ Wb}, \quad I = 25 \text{ A}$

$$\therefore \quad L = \frac{500\times50\times10^{-3}}{125} = \mathbf{0.2\ H}$$

$$v = -L\frac{dI}{dt}$$

$$= -L\left[\frac{\text{Final value of I} - \text{Initial value of I}}{\text{Time}}\right]$$

$$\therefore \quad v = -0.2\times\left(\frac{0-125}{0.1}\right) = \mathbf{250\ volts}$$

This is positive because current is decreased. So this 'v' will try to oppose this decrease, means will try to increase current and will help the growth of the current.

Mutually Induced E.M.F. and Mutual Inductance (M)

If the flux produced by one coil links with the other coil, placed sufficiently close to the first coil, then due to the change in the flux produced by first coil, there is induced e.m.f. in second coil. Such induced e.m.f. in the second coil is called **mutually induced e.m.f.**

Consider two coils which are placed very close to each other as shown in the Fig.

Let coil 1 has N_1 turns, while coil 2 has N_2 turns. The current flowing through coil 1 is i_1. Due to this current, the flux produced in coil 1 is ϕ_1. The part of this flux links with coil 2. This flux is called mutual flux.

It is denoted by ϕ_{21} as it is a part of flux ϕ_1 linking with coil 2. When current through coil 1 changes, the flux produced in coil 1 i.e. ϕ_1 changes. Thus flux associated with coil 2 i.e. ϕ_{21} changes. So according to the Faraday's law, there will be induced e.m.f. in coil 2.

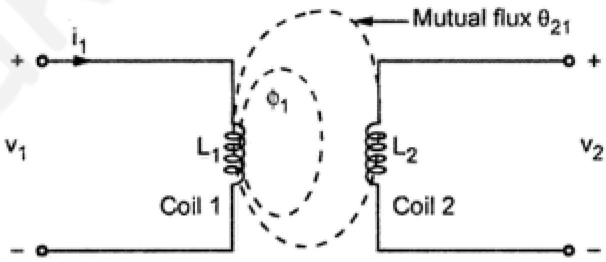


Fig.

Magnitude of Mutually Induced E.M.F.

Let N_1 = Number of turns of coil 1

N_2 = Number of turns of coil 2

I_1 = Current flowing through coil 1

ϕ_1 = Flux produced due to current I_1 in webers.

ϕ_2 = Flux linking with coil 2

According to Faraday's law, the induced e.m.f. in coil B is,

$$v_2 = -N_2\frac{d\phi_2}{dt}$$

Negative sign indicates that this e.m.f. will set up a current which will oppose the change of flux linking with it.

Now $$\phi_2 = \frac{\phi_2}{I_1} \times I_1$$

If permeability of the surroundings is assumed constant then $\phi_2 \propto I_1$ and hence ϕ_2 / I_1 is constant.

$$\therefore \quad \text{Rate of change of } \phi_2 = \frac{\phi_2}{I_1} \times \text{Rate of change of current } I_1$$

$$\therefore \quad \frac{d\phi_2}{dt} = \frac{\phi_2}{I_1} \cdot \frac{dI_1}{dt}$$

$$\therefore \quad v_2 = -N_2 \cdot \frac{\phi_2}{I_1} \cdot \frac{dI_1}{dt}$$

$$\therefore \quad v_2 = -\left(\frac{N_2 \phi_2}{I_1}\right)\frac{dI_1}{dt}$$

Here $\left(\frac{N_2 \phi_2}{I_1}\right)$ is called coefficient of mutual inductance denoted by M.

$$\therefore \quad \boxed{v_2 = -M\frac{dI_1}{dt} \quad \text{volts}}$$

Coefficient of mutual inductance is defined as the property by which e.m.f. gets induced in the second coil because of change in current through first coil.

Coefficient of mutual inductance is also called mutual inductance. It is measured in henries.

Coefficient of Coupling or Magnetic Coupling Coefficient (k)

Consider two coils having self inductances L_1 and L_2 placed very close to each other. Let the number of turns of the two coils be N_1 and N_2 respectively. Let coil 1 carries current i_1 and coil 2 carries current i_2.

Due to current i_1, the flux produced is ϕ_1 which links with both the coils. Then from the previous knowledge mutual inductance between two coils can be written as

$$M = \frac{N_1 \phi_{21}}{i_1} \quad \text{... (1)}$$

where ϕ_{21} is the part of the flux ϕ_1 linking with coil 2. Hence we can write, $\phi_{21} = k_1 \phi_1$.

$$\therefore \quad M = \frac{N_1 (k_1 \phi_1)}{i_1} \quad \text{... (2)}$$

Similarly due to current i_2, the flux produced is ϕ_2 which links with both the coils. Then the mutual inductance between two coils can be written as

$$M = \frac{N_2 \phi_{12}}{i_2} \quad \text{... (3)}$$

where ϕ_{12} is the part of the flux ϕ_2 linking with coil 1. Hence we can write $\phi_{12} = k_2 \phi_2$.

$$\therefore \quad M = \frac{N_2 (k_2 \phi_2)}{i_2} \quad \text{... (4)}$$

Multiplying equations (2) and (4),

$$M^2 = \frac{N_1 (k_1 \phi_1)}{i_1} \cdot \frac{N_2 (k_2 \phi_2)}{i_2}$$

$$\therefore \quad M^2 = k_1 k_2 \left[\frac{N_1 \phi_1}{i_1}\right]\left[\frac{N_2 \phi_2}{i_2}\right]$$

But

$$\frac{N_1\phi_1}{i_1} = \text{Self inductance of coil 1} = L_1$$

$$\frac{N_2\phi_2}{i_2} = \text{Self inductance of coil 2} = L_2$$

$$\therefore \quad M^2 = k_1 k_2 L_1 L_2$$

$$\therefore \quad M = \sqrt{k_1 k_2}\sqrt{L_1 L_2}$$

Let $k = \sqrt{k_1 k_2}$

$$\therefore \quad \boxed{M = k\sqrt{L_1 L_2}} \quad \text{... (5)}$$

where k is called **coefficient of coupling.**

$$\therefore \quad \boxed{k = \frac{M}{\sqrt{L_1 L_2}}} \quad \text{... (6)}$$

Example 2 : *The number of turns in two coupled coils are 600 and 1200 respectively. When a current of 4 A flows in coil 1, the total flux in coil 1 is 0.5 mWb and the flux linking coil 2 is 0.4 mWb. Determine the self inductances of the coils and mutual inductance between them. Also calculate coefficient of coupling.*

Solution :

For coil 1, $N_1 = 600$

$i_1 = 4 \text{ A}$

$\phi_1 = 0.5 \text{ mWb}$

$$\therefore \quad L = \frac{N_1\phi_1}{i_1} = \frac{(600)(0.5\times 10^{-3})}{4} = \mathbf{0.075\ H}$$

The self inductance of a coil is directly proportional to the square of the number of turns i.e. $L \propto N^2$.

$$\therefore \quad \frac{L_1}{L_2} = \frac{N_1^2}{N_2^2}$$

$$\therefore \quad L_2 = \left(\frac{N_2}{N_1}\right)^2 \cdot L_1 = \left(\frac{1200}{600}\right)^2 (0.075) = \mathbf{0.3\ H}$$

The flux linking with coil 2 is $\phi_{22} = 0.4$ mWb

$$\therefore \quad M = \frac{N_2\phi_{21}}{i_1}$$

$$= \frac{(1200)\times(0.4\times 10^{-3})}{4}$$

$$= \mathbf{0.12\ H}$$

Hence the coefficient of coupling is given by,

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.12}{\sqrt{(0.075)(0.3)}} = \mathbf{0.8}$$

Dot Conventions

The sign of mutually induced voltage depends on direction of winding of the coils. But it is very inconvenient to supply the information about winding direction of the coils. Hence dot conventions are used for purpose of indicating direction of winding. The dot conventions are interpreted as below :

1. **If a current enters a dot in one coil, then mutually induced voltage in other coil is positive at the dotted end.**
2. **If a current leaves a dot in one coil, then mutually induced voltage in other coil is negative at the dotted end.**

Consider two magnetically coupled coils L_1 and L_2 wound on same core. Let current through coils L_1 and L_2 be i_1 and i_2 respectively. All the possible combinations of the dot convention between the magnetically coupled coils are as shown in the Fig. 2.4 (a), (c), (e) and (g). The equivalent circuits of all possible dot convention are as shown in the Fig. 2.4 (b), (d), (f) and (h) respectively.

Consider a magnetically coupled circuit with dots placed as shown in the Fig. 2.4 (a). Both the currents, i_1 and i_2 are entering the dotted terminals. Hence according to the dot convention, the mutually induced e.m.f. in both the coils has the polarity same as self

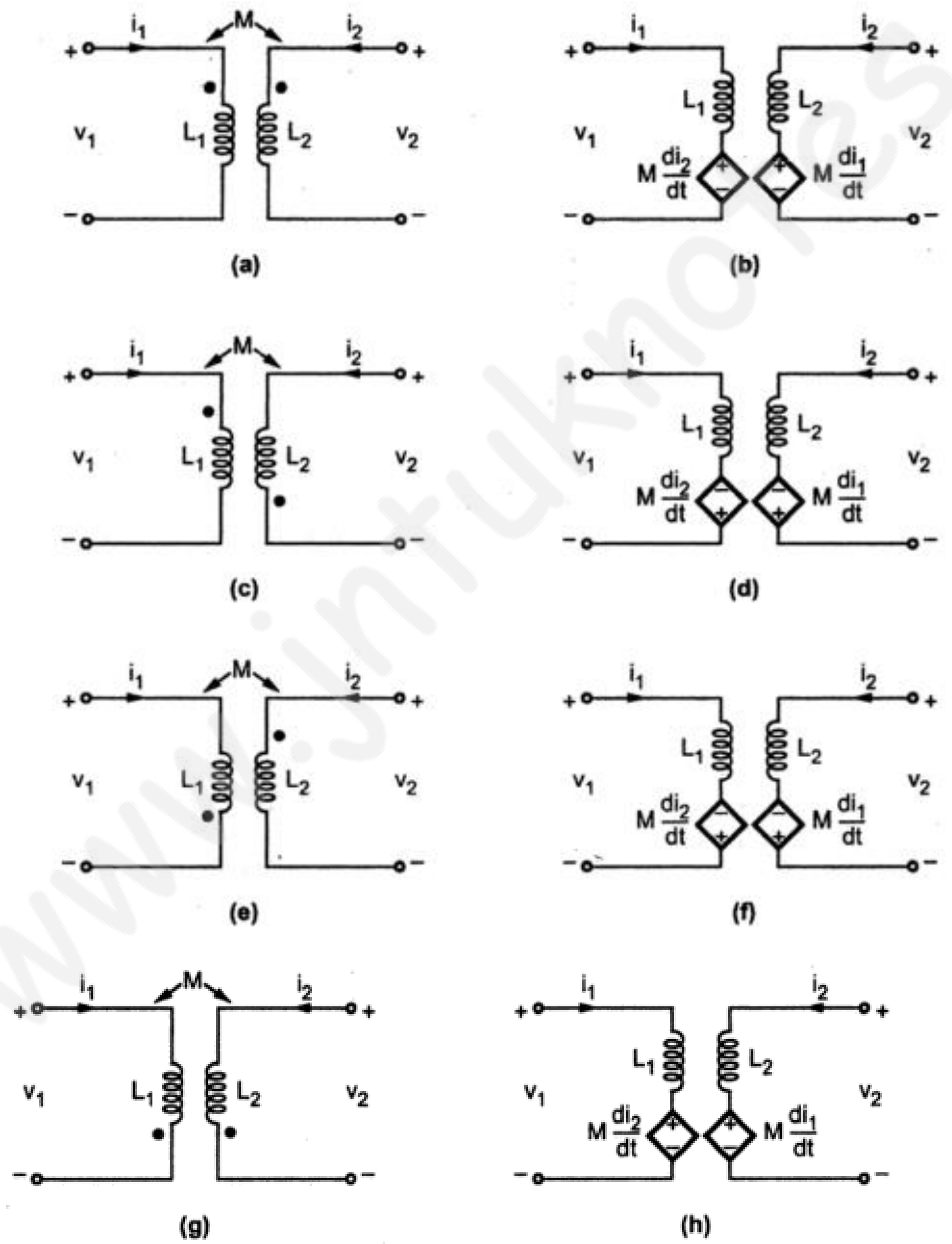


Fig. 2.4 Magnetically coupled circuits and equivalent circuits with different dot conventions

induced e.m.f. in respective coil. The equivalent circuit is as shown in the Fig. 2.4 (b). Applying KVL, the network equations of the equivalent circuit can be written as :

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \quad \text{... (1)}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \quad \text{... (2)}$$

Now consider magnetically coupled circuit as shown in the Fig. 2.4 (c) with dot placed at lower terminal of coil L_2. Hence current i_1 enters through dotted terminal of L_1 while current i_2 leaves through dotted terminal of L_2. So according to dot convention, the polarity of mutually induced e.m.f. in L_1 due to i_2 in L_2 will be opposite to that of self induced e.m.f. in coil L_1. Also the polarity of mutually induced e.m.f. in coil L_2 due to the current i_1 in coil L_1 will be opposite to that of self induced e.m.f. in coil L_2. The equivalent circuit is as shown in the Fig. 2.4 (d). By using KVL, the network equations can be written as,

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \quad \text{... (3)}$$

$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \quad \text{... (4)}$$

For the equivalent circuit shown in the Fig. 2.4 (f). Applying KVL, the network equations can be written as,

$$v_1 = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \quad \text{... (5)}$$

$$v_2 = L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} \quad \text{... (6)}$$

For last possible combination, both the dots are placed at lower terminals of coils L_1 and L_2. Also both the currents leave dot as shown in the Fig. 2.4 (g). The equivalent circuit is as shown in the Fig. 2.4 (h). By applying KVL, the network equations can be written as,

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \quad \text{... (7)}$$

$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \quad \text{... (8)}$$

Uptill now we have discussed the coupled circuits in which two coils are magnetically coupled. But practically we may have to analyze a network with several windings.

The analysis of multiwinding inductor networks can be carried out for each pair of windings using same dot convention. In case of multiwinding inductor networks, the relationship between each pair of windings is represented by different forms of the dots such as ■ , ▲ , ● , ★ etc. The analysis of such multiwinding networks is illustrated in Example 2.4 and Example 2.5.

➡ **Example 3 :** *Calculate effective inductance of the circuit shown (Fig. 2.5) across terminals a and b.*

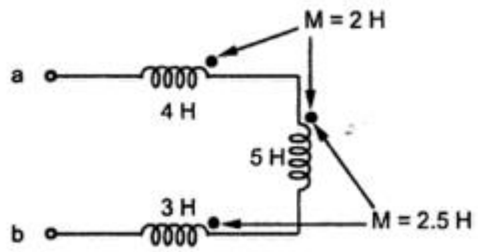


Fig. 2.5

Solution : Assume that current 'i' is flowing in series circuit and voltage developed across terminals a and b is shown in following Fig. 2.5 (a).

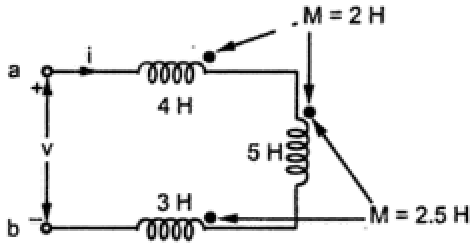


Fig. 2.5 (a)

Applying KVL for the above circuit. The current flowing through all the coils is same i.e. 'i'.

While writing the equations follow the convention that the current entering in the dot of one coil produces positive at the dotted end of the another coil while the current leaving from the dotted end of one coil produces the negative at the dotted end of the another coil.

$$-4\frac{di}{dt}+2\frac{di}{dt}-5\frac{di}{dt}+2\frac{di}{dt}-2.5\frac{di}{dt}-3\frac{di}{dt}-2.5\frac{di}{dt}+v=0$$

$$\therefore \quad v = 13\frac{di}{dt} = L_{eff}\frac{di}{dt}$$

$\therefore$ Effective inductance across terminals a and b is L_{eff}.

$$\therefore \quad L_{eff} = \mathbf{13\ H}$$

Inductive Coupling in Series

When two inductors having self inductances L_1 and L_2 are coupled in series, mutual inductance M exists between them. Two kinds of series connection are possible as follows.

Series Aiding

In this connection, two coils are connected in series such that their induced fluxes or voltages are additive in nature.

Here currents i_1 and i_2 is nothing but current i which is entering dots for both the coils.

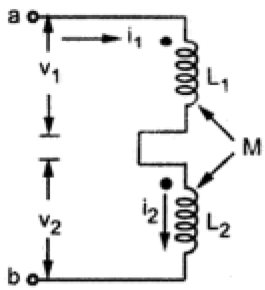


Fig. 2.8

Self induced voltage in coil 1 = $v_1 = -L_1\frac{di}{dt}$

Self induced voltage in coil 2 = $v_2 = -L_2\frac{di}{dt}$

Mutually induced voltage in coil 1 due to change in current in coil 2 = $v_1' = -M\frac{di}{dt}$

Mutually induced voltage in coil 2 due to change in current in coil 1 = $v_2' = -M\frac{di}{dt}$

$\therefore$ Total induced voltage = $v_1 + v_2 + v_1' + v_2'$

$$= -\left(L_1\frac{di}{dt}+L_2\frac{di}{dt}+M\frac{di}{dt}+M\frac{di}{dt}\right)$$

$$= -(L_1+L_2+2M)\frac{di}{dt}$$

If L is equivalent inductance across terminals a-b then total induced voltage in single inductance would be equal to $-L_{eff}\frac{di}{dt}$. Comparing two voltages,

$$\boxed{L_{eff} = L_1 + L_2 + 2M}$$

Series Opposing

In this connection, two coils are connected in such a way that, the induced fluxes or voltages are of opposite polarities.

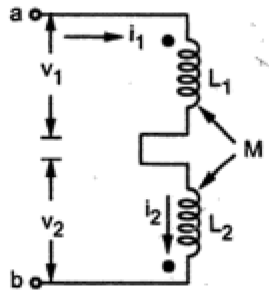


Fig. 2.9

Here i_1 and i_2 is same series current 'i' which is entering dot for coil L_1 and leaving dot for coil L_2.

Self induced voltage in coil 1 $= -L_1\frac{di}{dt}$

Self induced voltage in coil 2 $= -L_2\frac{di}{dt}$

Mutually induced voltage in coil 1 due to change in current in coil 2 $= v_1' = +M\frac{di}{dt}$

Also Mutually induced voltage in coil 2 due to change in current in coil 1 $= v_2' = +M\frac{di}{dt}$

Therefore total induced voltage $= v_1 + v_2 + v_1' + v_2'$

$$= -L_1\frac{di}{dt} - L_2\frac{di}{dt} + M\frac{di}{dt} + M\frac{di}{dt}$$

$$= -(L_1 + L_2 - 2M)\frac{di}{dt}$$

If L is equivalent inductance across terminals a and b then total induced voltage in single inductance would be equal to $-L_{eff}\frac{di}{dt}$. Comparing two voltages,

$$L_{eff} = L_1 + L_2 - 2M$$

Inductive Coupling in Parallel

When two inductors having self inductances L_1 and L_2 are coupled in parallel, we have two kinds of connections as follows.

Parallel Aiding

Consider parallel coupling of two inductors as shown in Fig. 2.10.

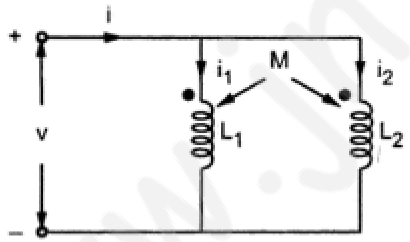


Fig. 2.10

Applying Kirchhoff's voltage law to both loops, we get,

$$-j\omega L_1 i_1 - j\omega M i_2 + v = 0$$

$$-j\omega L_2 i_2 - j\omega M i_1 + v = 0$$

i.e. $v = j\omega L_1 i_1 + j\omega M i_2$ ... (1)

$v = j\omega L_2 i_2 + j\omega M i_1$ ... (2)

We have, $j\omega L_1 \cdot i_1 + j\omega M \cdot i_2 = j\omega L_2 \cdot i_2 + j\omega M \cdot i_1$

But $i = i_1 + i_2$

i.e. $i_2 = i - i_1$

Putting value of i_2 in above equation, we get

$\therefore \quad j\omega L_1 i_1 + j\omega M(i - i_1) = j\omega L_2 (i - i_1) + j\omega M i_1$

$\therefore \quad j\omega i_1(L_1 + L_2 - 2M) = j\omega i(L_2 - M)$

$$\therefore \qquad i_1 = \left[\frac{L_2 - M}{L_1 + L_2 - 2M}\right] i$$

Similarly,

$$i_2 = \left[\frac{L_1 - M}{L_1 + L_2 - 2M}\right] i$$

Putting values of i_1 and i_2 in equation (1), we get,

$$v = j\omega\left[\frac{L_1 (L_2 - M)}{L_1 + L_2 - 2M} + \frac{M (L_1 - M)}{L_1 + L_2 - 2M}\right] i$$

$$\therefore \qquad v = j\omega\left[\frac{L_1 L_2 - L_1 M + L_1 M - M^2}{L_1 + L_2 - 2M}\right] i$$

$$\therefore \qquad v = j\omega\left[\frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}\right] i \qquad \ldots (3)$$

If L is effective inductance of parallel combination then,

$$v = j\omega L_{eff} \cdot i \qquad \ldots (4)$$

Comparing equations (3) and (4) we have

$$\boxed{L_{eff} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}}$$

Parallel Opposing

Consider two inductors connected in parallel as shown in Fig. 2.11.

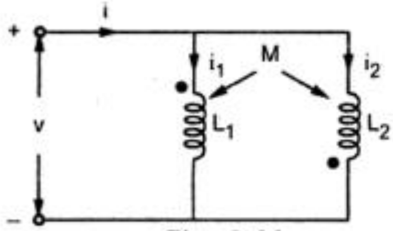


Fig. 2.11

Applying KVL to both loops, we get,

$$-j\omega L_1 i_1 + j\omega M i_2 + v = 0$$

$$-j\omega L_2 i_2 + j\omega M i_1 + v = 0$$

i.e. $j\omega L_1 i_1 - j\omega M i_2 = v \qquad \ldots (5)$

$$j\omega L_2 i_2 - j\omega M i_1 = v \qquad \ldots (6)$$

We have, $\quad j\omega L_1 i_1 - j\omega M i_2 = j\omega L_2 i_2 - j\omega M i_1$

But $\quad i = i_1 + i_2$

$$\therefore \qquad i_2 = i - i_1$$

Substituting value of i_2 in above equation we have,

$$j\omega L_1 i_1 + j\omega M (i - i_1) = j\omega L_2 (i - i_1) - j\omega M i_1$$

$$\therefore \qquad j\omega i_1 (L_1 + L_2 + 2M) = j\omega i (L_2 + M)$$

$$\therefore \qquad i_1 = \left[\frac{L_2 + M}{L_1 + L_2 + 2M}\right] i$$

Similarly,

$$i_2 = \left[\frac{L_1 + M}{L_1 + L_2 + 2M}\right] i$$

Putting values of i_1 and i_2 in equation (5) we get,

$$v = j\omega\left[\frac{L_1 (L_2 + M)}{L_1 + L_2 + 2M} - \frac{M (L_1 + M)}{L_1 + L_2 + 2M}\right] i$$

$$= j\omega\left[\frac{L_1 L_2 + L_1 M - L_1 M - M^2}{L_1 + L_2 + 2M}\right] i$$

$$= j\omega\left[\frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}\right] i \qquad \ldots (7)$$

If L is effective inductance of parallel combination,

$$v = j\omega L_{eff} \cdot i \qquad \dots (8)$$

Comparing equations (7) and (8) we have,

$$\boxed{L_{eff} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}}$$

Example : *If a coil of 800 μH is magnetically coupled to another coil of 200 μH. The coefficient of coupling between two coils is 0.05. Calculate inductance if two coils are connected in,*

(i) Series aiding (ii) Series opposing (iii) Parallel aiding (iv) Parallel opposing

Solution : The mutual inductance between two coils is given by

$$M = k \cdot \sqrt{L_1 L_2} = (0.05)\sqrt{(800 \times 10^{-6})(200 \times 10^{-6})}$$

$$= 20\ \mu H$$

Let the effective inductance for magnetically coupled coil be L.

(i) Series aiding :

$$L = L_1 + L_2 + 2M$$

$$= (800 \times 10^{-6}) + (200 \times 10^{-6}) + (2 \times 20 \times 10^{-6})$$

$$= \mathbf{1040\ \mu H}$$

(ii) Series opposing :

$$L = L_1 + L_2 - 2M$$

$$= (800 \times 10^{-6}) + (200 \times 10^{-6}) - (2 \times 20 \times 10^{-6})$$

$$= \mathbf{960\ \mu H}$$

(iii) Parallel aiding :

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$= \frac{(800 \times 10^{-6} \times 200 \times 10^{-6}) - (20 \times 10^{-6})^2}{960 \times 10^{-6}}$$

$$= \frac{0.1596 \times 10^{-6}}{960 \times 10^{-6}} = \mathbf{166.25\ \mu H}$$

(iv) Parallel opposing :

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

$$= \frac{(800 \times 10^{-6} \times 200 \times 10^{-6}) - (20 \times 10^{-6})^2}{1040 \times 10^{-6}}$$

$$= \frac{0.1596 \times 10^{-6}}{1040 \times 10^{-6}}$$

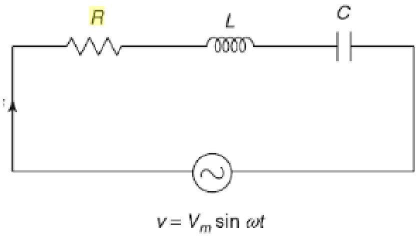
$$= \mathbf{153.46\ \mu H}$$

RESONANCE

Introduction:

Electrical resonance occurs in an electric circuit at a particular *resonant frequency* when the impedances or admittances of circuit elements cancel each other. In some circuits, this happens when the impedance between the input and output of the circuit is almost zero

SERIES RESONANCE



Series circuit

Resonance is a very important phenomenon in many electrical applications. The study of resonance is very useful in the telecommunication field. A circuit containing reactance is said to be in resonance if the voltage across the circuit is in phase with the current through it. At resonance, the circuit thus behaves as a pure resistor and the net reactance is zero.

As $X_L = 2\pi fL$. As frequency is changed from 0 to ∞, X_L increases linearily and graph of X_L against f is straight line passing through origin.

As $X_C = \frac{1}{2\pi fC}$, as frequency is changed from 0 to ∞, X_C reduces and the graph of X_C against f is rectangular hyperbola. Mathematically sign of X_C is opposite to X_L hence graph of X_L Vs f is shown in the first quadrant while X_C Vs f is shown in the third quadrant.

At $f = f_r$, the value of $X_L = X_C$ at this frequency.

As $X = X_L - X_C$, the graph of X against f is shown in the Fig. 4.1.

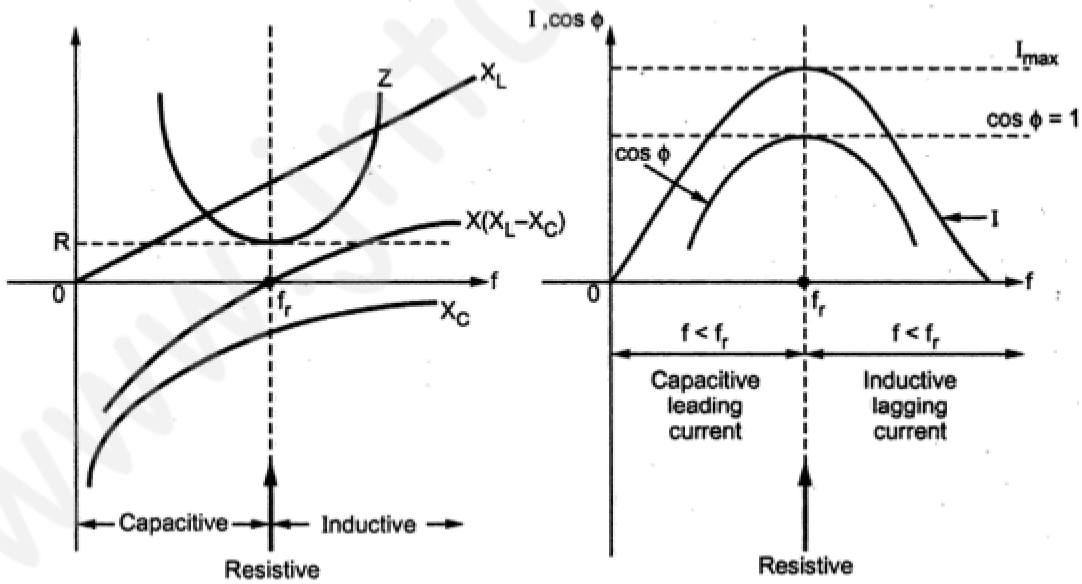


Fig. 4.1 Characteristics of series resonance

For $f < f_r$, the $X_C > X_L$ and net reactance X is **capacitive** while for $f > f_r$, the $X_L > X_C$ and net reactance X is **inductive**.

Now $Z = R + jX = R + j(X_L - X_C)$ but at $f = f_r$, $X_L = X_C$ and $X = 0$ hence the net impedance $Z = R$ which is purely resistive. So **impedance is minimum and purely resistive** at series resonance. The graph of Z against f is also shown in the Fig. 4.1.

Key Point : *As impedance is minimum, the current I = V/Z is maximum at series resonance.*

Now power factor cos $\phi = R/Z$ and at $f = f_r$ as $Z = R$, the **power factor is unity** and at its **maximum** at series resonance. For $f < f_r$ it is **leading** in nature while for $f > f_r$ it is **lagging** in nature.

Resonant Frequency

Let f_r be the resonant frequency in Hz at which,

$$X_L = X_C$$

$$\therefore \quad 2\pi f_r L = \frac{1}{2\pi f_r C}$$

$$\therefore \quad (f_r)^2 = \frac{1}{4\pi^2 LC}$$

$$\therefore \quad f_r = \frac{1}{2\pi\sqrt{LC}} \text{ Hz}$$

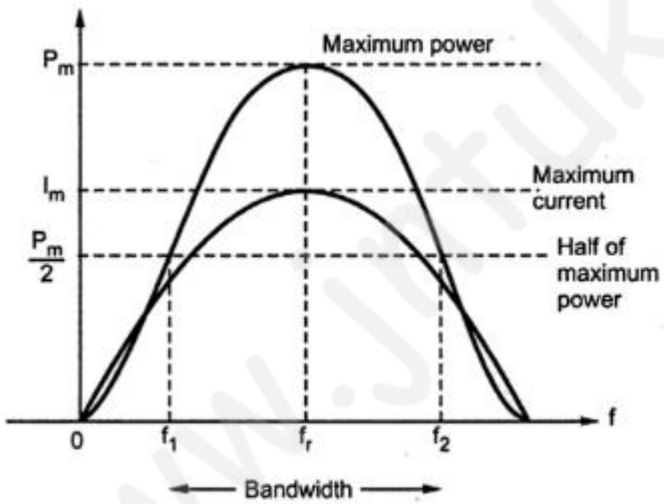
i.e.

$$\boxed{\omega_r = \frac{1}{\sqrt{LC}} \text{ rad/sec}}$$

Bandwidth of Series R-L-C Circuit

At series resonance, current is maximum and impedance Z is minimum. Now power consumed in a circuit is proportional to square of the current as $P = I^2R$. So at series resonance as current is maximum, power is also at its maximum i.e. P_m. The Fig. 4.2 shows the graph of current and power against frequency.

It can be observed that at two frequencies f_1 and f_2 the power is half of its maximum value. These frequencies are called **half power frequencies.**



The difference between the half power frequencies f_1 and f_2 at which power is half of its maximum is called bandwidth of the series R-L-C circuit.

$\therefore$ B.W. = $f_2 - f_1$

Expressions for Lower and Upper Cut-off Frequencies

The current in a series RLC circuit is given by the equation,

$$I = \frac{V}{Z} \qquad \text{but } Z = R + j\,(X_L - X_C)$$

$$\therefore \quad I = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad \text{... (1)}$$

At resonance, $I_m = \frac{V}{R}$ (maximum value) ... (2)

and $P_m = I_m^2 R$

At half power point, $P = \frac{P_m}{2} = \frac{I_m^2}{2} R = \left(\frac{I_m}{\sqrt{2}}\right)^2 R$

$$\therefore \quad I = \frac{I_m}{\sqrt{2}} \quad \text{at half power frequency}$$

Equating equations (1) and (2),

$$\frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V}{\sqrt{2} \cdot R}$$

$$\therefore \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}\, R$$

$$\therefore \quad R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2\,R^2$$

$$\therefore \quad \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\therefore \quad \omega L - \frac{1}{\omega C} = \pm R \qquad \text{...(3)}$$

From the equation (3) we can find two values of half power frequencies which are ω_1 and ω_2 corresponding to f_1 and f_2.

$$\therefore \quad \omega_2 L - \frac{1}{\omega_2 C} = +R \qquad \text{...(4)}$$

$$\text{and} \quad \omega_1 L - \frac{1}{\omega_1 C} = -R \qquad \text{... (5)}$$

$$\therefore \quad (\omega_1 + \omega_2)\, L - \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\frac{1}{C} = 0 \qquad \text{... Adding equations (4) and (5)}$$

$$\therefore \quad (\omega_1 + \omega_2)\, L = \frac{(\omega_1 + \omega_2)}{\omega_1\, \omega_2} \cdot \frac{1}{C}$$

$$\therefore \quad \omega_1\, \omega_2 = \frac{1}{LC} \qquad \text{... (6)}$$

$$\text{but} \quad \omega_r = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad \omega_1\, \omega_2 = (\omega_r)^2$$

$$\therefore \quad f_1 f_2 = (f_r)^2 \qquad \text{... (7)}$$

The equation (7) shows that the **resonant frequency is the geometric mean of the two half power frequencies.**

$$\therefore \qquad \boxed{f_r = \sqrt{f_1 f_2}} \qquad \dots (8)$$

Subtracting equation (5) from equation (4) we get,

$$(\omega_2 - \omega_1) L - \left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right)\frac{1}{C} = 2R$$

$$\therefore \qquad (\omega_2 - \omega_1) + \frac{(\omega_2 - \omega_1)}{\omega_1 \omega_2} \cdot \frac{1}{LC} = \frac{2R}{L} \qquad \dots \text{Dividing both sides by L}$$

$$\therefore \qquad (\omega_2 - \omega_1) + (\omega_2 - \omega_1) = \frac{2R}{L} \qquad \dots \text{As } \frac{1}{\omega_1 \omega_2} = LC$$

$$\therefore \qquad (\omega_2 - \omega_1) = \frac{R}{L}$$

$$\text{i.e.} \qquad f_2 - f_1 = \frac{R}{2\pi L} \qquad \dots (9)$$

$$\text{Thus} \qquad \boxed{B.W. = \frac{R}{2\pi L}}$$

The bandwidth is also denoted as,

B.W. = $2\Delta f$ where

$$\boxed{\Delta f = \frac{R}{4\pi L}}$$ as shown in the Fig. 4.3

From Fig. 4.3 we can write,

$$f_1 = f_r - \Delta f$$

and $$f_2 = f_r + \Delta f$$

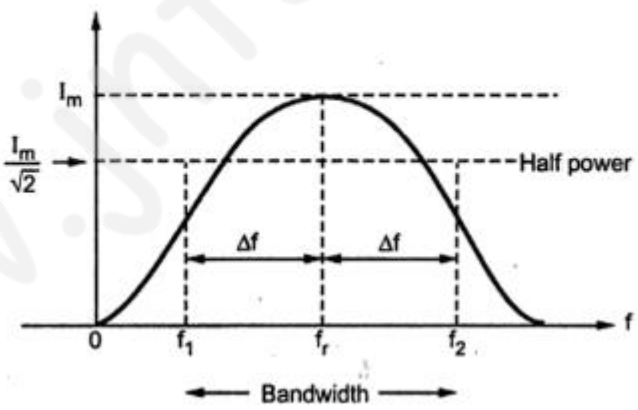


Fig. 4.3

Quality Factor

The quality factor of R-L-C series circuit is the voltage magnification in the circuit at resonance.

$$\text{Voltage magnification} = \frac{\text{Voltage across L or C}}{\text{Supply voltage}}$$

Now $\quad V_L = \text{Voltage across } L = I_m X_L = I_m \omega_r L$ at resonance

and $\quad I_m = \frac{V}{R}$ at resonance

$\therefore \quad V_L = \frac{V\omega_r L}{R}$ at resonance

$\therefore \quad \text{Voltage magnification} = \frac{\frac{V\omega_r L}{R}}{V} = \frac{\omega_r L}{R}$

This is nothing but quality factor Q.

$\therefore \quad \boxed{Q = \frac{\omega_r L}{R}}$ $\quad$ but $\omega_r = \frac{1}{\sqrt{LC}}$

$\therefore \quad \boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$

and $\quad \boxed{Q = \frac{\omega_r}{B.W.}}$ $\quad$ as B.W.= $(\omega_2 - \omega_1) = \frac{R}{L}$

Example 1 : *A RLC series circuit with a resistance of 10 Ω, impedance of 0.2 H and a capacitance of 40μF is supplied with a 100 V supply at variable frequency. Find the following w.r.t the series resonant circuit :-*

i) the frequency at resonance ii) the current iii) power iv) power factor v) voltage across R, L, C at that time vi) quality factor of the circuit vii) half power points viii) phasor diagram.

Solution : The given values are, R = 10 Ω,, L = 0.2 H, C = 40 μF and V = 100 V

i) $\quad f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.2\times 40\times 10^{-6}}}$

$\quad = \mathbf{56.2697\ Hz}$

ii) $\quad I_m = \frac{V}{R} = \frac{100}{10} = \mathbf{10\ A}$ $\quad$ ... Current is maximum at resonance

iii) $\quad P_m = I_m^2 R = (10)^2 \times 10 = \mathbf{1000\ W}$

iv) Power factor is **unity**, as impedance is **purely resistive** at resonance

v) $\quad V_R = I_m R = 10 \times 10 = \mathbf{100\ V}$

$\quad X_L = 2\pi f_r L = 2\pi \times 56.2697 \times 0.2 = 70.7105\ \Omega$

$\therefore \quad V_L = I_m X_L = 10 \times 70.7105 = \mathbf{707.105\ V}$

and $\quad X_C = \frac{1}{2\pi f_r C}$

$\quad = \frac{1}{2\pi \times 56.2697 \times 40 \times 10^{-6}}$

$\quad = 70.7105\ \Omega$

$$\therefore \quad \boxed{V_C = I_m X_C = 707.105 \text{ V}}$$

Thus $V_L = V_C$ at resonance

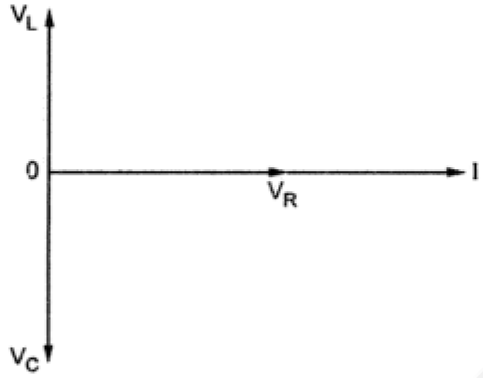
vi) $$Q = \frac{\omega_r L}{R} = \frac{2\pi f_r L}{R} = 7.071$$

vii) $$\Delta f = \frac{R}{4\pi L} = \frac{10}{4\pi \times 0.2} = 3.9788$$

$$\therefore \quad f_1 = f_r - \Delta f = 56.2697 - 3.9788 = 52.2909 \text{ Hz}$$

and $$f_2 = f_r + \Delta f = 56.2697 + 3.9788 = 60.2485 \text{ Hz}$$

viii) $$\text{B.W.} = f_2 - f_1 = 60.2485 - 52.2909 = 7.9576 \text{ Hz}$$



Resonance in Parallel Circuit

Similar to a series a.c. circuit, there can be a resonance in parallel a.c. circuit. When the power factor of a parallel a.c. circuit is unity i.e. the voltage and total current are in phase at a particular frequency then the parallel circuit is said to be at **resonance**. The frequency at which the parallel resonance occurs is called **resonant frequency** denoted as f_r Hz.

4.3.1 Characteristics of Parallel Resonance

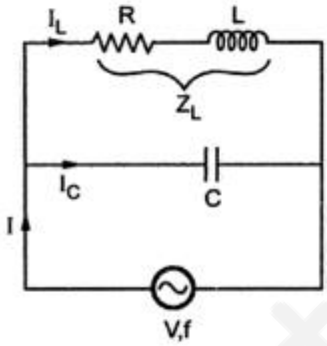


Fig. 4.5 Practical parallel circuit

Consider a practical parallel circuit used for the parallel resonance as shown in the Fig. 4.5.

The one branch consists of resistance R in series with inductor L. So it is series R-L circuit with impedance Z_L. The other branch is pure capacitive with a capacitor C. Both the branches are connected in parallel across a variable frequency constant voltage source.

The current drawn by inductive branch is I_L while drawn by capacitive branch is I_C.

$$I_L = \frac{V}{Z_L} \quad \text{where } Z_L = R + j X_L$$

and $$I_C = \frac{V}{X_C} \quad \text{where } X_C = \frac{1}{2\pi f C}$$

The current I_L lags voltage V by angle ϕ_L which is decided by R and X_L while the current I_C leads voltage V by 90°. The total current I is phasor addition of I_L and I_C. The phasor diagram is shown in the Fig. 4.5 (a).

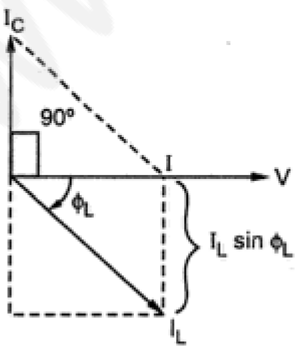


Fig. 4.5(a)

For the parallel resonance V and I must be in phase. To achieve this unity p.f. condition,

$$I = I_L \cos \phi_L$$

and
$$I_C = I_L \sin \phi_L$$

From the impedance triangle of R-L series circuit we can write,

$\tan \phi_L = \frac{X_L}{R}$, $\cos \phi_L = \frac{R}{Z_L}$, $\sin \phi_L = \frac{X_L}{Z_L}$

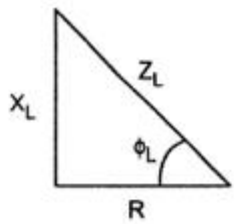


Fig. 4.5(b) Impedance triangle

As frequency is increased, $X_L = 2\pi f L$ increases due to which $Z_L = \sqrt{R^2 + X_L^2}$ also increases. Hence $\cos \phi_L$ decreases and $\sin \phi_L$ increases. As Z_L increases, the current I_L also decreases.

At resonance $f = f_r$ and $I_L \cos \phi_L$ is at its minimum. Thus at resonance **current is minimum** while the total **impedance** of the circuit is **maximum**. As admittance is reciprocal of impedance, as frequency is changed, **admittance** decreases and is **minimum** at resonance. The three curves are shown in the Fig. 4.6 (a), (b) and (c).

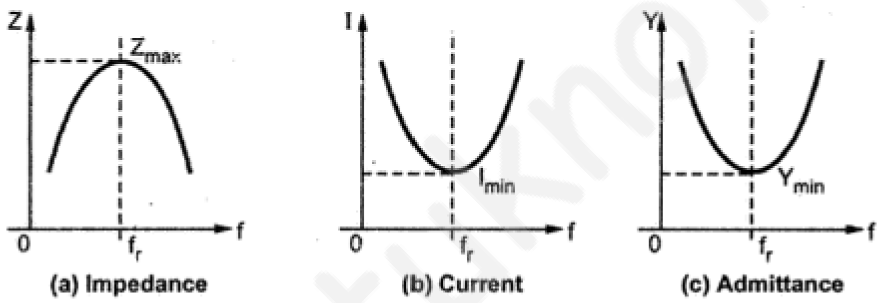


Fig. 4.6 Characteristics of parallel resonance

Expression for Resonant Frequency

At resonance $I_C = I_L \sin \phi_L$

$$\therefore \quad \frac{V}{X_C} = \frac{V}{Z_L} \cdot \frac{X_L}{Z_L} = \frac{V X_L}{Z_L^2}$$

$$\therefore \quad Z_L^2 = X_L X_C$$

$$\therefore \quad R^2 + (2\pi f_r L)^2 = (2\pi f_r L) \times \frac{1}{2\pi f_r C} \quad \text{as } f = f_r$$

$$\therefore \quad R^2 + (2\pi f_r L)^2 = \frac{L}{C}$$

$$\therefore \quad (2\pi f_r L)^2 = \frac{L}{C} - R^2$$

$$\therefore \quad (2\pi f_r)^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\therefore \quad f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

Thus if R is very small compared to L and C, $\frac{R^2}{L^2} << \frac{1}{LC}$

$$\therefore \quad f_r = \frac{1}{2\pi\sqrt{LC}}$$

... Neglecting effect of R

Dynamic Impedance at Resonance

The impedance offered by the parallel circuit at resonance is called **dynamic impedance** denoted as Z_D. This is maximum at resonance. As current drawn at resonance is minimum, the parallel circuit at resonance is called **rejector circuit**. This indicates that it rejects the unwanted frequencies and hence it is used as filter in radio receiver.

From $I_C = I_L \sin \phi_L$ we have seen that,

$$Z_L^2 = \frac{L}{C}$$

While $$I = I_L \cos \phi_L = \frac{V}{Z_L} \cdot \frac{R}{Z_L}$$

$$= \frac{VR}{Z_L^2}$$

$\therefore$ $$I = \frac{VR}{\frac{L}{C}}$$

$$= \frac{V}{(L/RC)}$$

$\therefore$ $$I = \frac{V}{Z_D}$$

where $$\boxed{Z_D = \frac{L}{RC} = \text{Dynamic impedance}}$$

Quality Factor of Parallel Circuit

The parallel circuit is used to magnify the current and hence known as current resonance circuit.

The quality factor of the parallel circuit is defined as the current magnification in the circuit at resonance.

The current magnification is defined as,

$$\text{Current magnification} = \frac{\text{Current in the inductive branch}}{\text{Current in supply at resonance}} = \frac{I_L}{I}$$

$$= \frac{\frac{V}{Z_L}}{\frac{V}{Z_D}} = \frac{Z_D}{Z_L}$$

$$= \frac{\frac{L}{RC}}{\sqrt{\frac{L}{C}}} = \frac{1}{R}\sqrt{\frac{L}{C}} \quad \text{as } Z_L$$

$$= \sqrt{X_L X_C} = \sqrt{\frac{L}{C}}$$

This is nothing but the quality factor at resonance.

$\therefore$ $$\boxed{Q = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

Example 2 : *An inductive coil of resistance 10 Ω and inductance 0.1 Henrys is connected in parallel with a 150 μF capacitor to a variable frequency, 200 V supply. Find the resonant frequency at which the total current taken from the supply is in phase with the supply voltage. Also find the value of this current. Draw the phasor diagram.*

Solution : The circuit is shown in the Fig. 4.7.

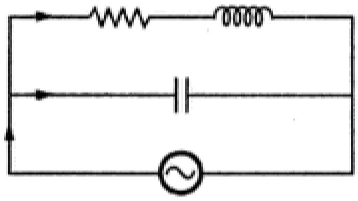


Fig. 4.7

The resonant frequency is,

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1 \times 150 \times 10^{-6}} - \frac{(10)^2}{(0.1)^2}}$$

$$= \mathbf{37.8865 \quad Hz}$$

Now $\quad Z_L = R + j\,X_L = 10 + j\;(2\pi f_r\, L)$

$$= 10 + j\,23.805 = 25.82\angle 67.21°\ \Omega$$

$\therefore \quad I_L = \frac{V}{Z_L} = \frac{200\angle 0°}{25.82\angle 67.21°} = \mathbf{7.7459\angle -67.21°\ A}$

and $\quad I_C = \frac{V}{X_C} = \frac{200\angle 0°}{\frac{1}{2\pi f_r C}\angle -90°} = \frac{200\angle 0°}{28\angle -90°} = \mathbf{7.143\angle +90°\ A}$

where $\quad Z_C = 0 - j\,X_C = 0 - j\,28 = 28\angle -90°\ \Omega$

$\therefore \quad Z_T = \frac{Z_C\, Z_L}{Z_C + Z_L} = \frac{28\angle -90° \times 25.82\angle 67.21°}{0 - j\,28 + 10 + j\,23.805}$

$$= \frac{722.96\angle -22.79°}{10 - j\,4.195}$$

$$= \frac{722.96\angle -22.79}{10.844\angle -22.79}$$

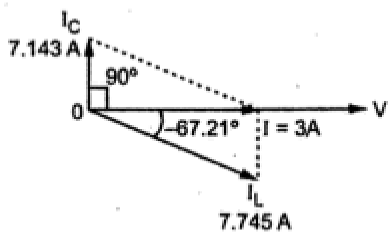
$$= \mathbf{66.67\ \Omega\ pure\ resistive}$$

$\therefore \quad Z_T = Z_D = \frac{L}{CR}$

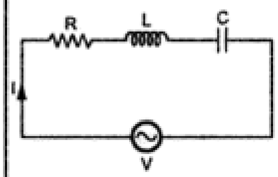
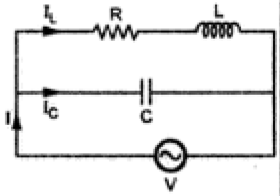
$$= \frac{0.1}{150 \times 10^{-6} \times 10}$$

$$= 66.67\ \Omega$$

$\therefore \quad I = \frac{V}{Z_D} = \frac{200}{66.67} = \mathbf{3\ A}$



Comparison of Resonant Circuits

Sr. No.	Parameter	Series Resonant	Parallel Resonant
1.	Circuit		
2.	Type of circuit	Purely resistive	Purely resistive
3.	Power factor	Unity	Unity
4.	Impedance	Minimum Z = R	Dynamic but maximum $Z_D = \frac{L}{RC}$
5.	Frequency	$f_r = \frac{1}{2\pi\sqrt{LC}}$	$f_r = \frac{1}{2\pi\sqrt{LC}}$
6.	Current	Maximum $I = \frac{V}{R}$	Minimum $I = \frac{V}{Z_D}$
7.	Magnification	Voltage magnification	Current magnification
8.	Quality factor	$Q = \frac{\omega_r L}{R} = \frac{\omega_r}{B.W.}$	$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$
9.	Nature	Acceptor	Rejector
10.	Practical used	Radio circuits sharpness of tunning circuit	Impedance for matching, tuning, as a filter